

# HARDY SPACES OF DIRICHLET SERIES AND PSEUDOMOMENTS OF THE RIEMANN ZETA FUNCTION

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**ABSTRACT.** We study  $H^p$  spaces of Dirichlet series, called  $\mathcal{H}^p$ , for  $0 < p < \infty$ . We begin by showing that  $\mathcal{H}^p$  may be defined either by taking an appropriate  $L^p$  closure of all Dirichlet polynomials or by requiring the sequence of “*mte Abschnitte*” to be uniformly bounded in  $L^p$ . After showing that these definitions are equivalent, we proceed to establish upper and lower weighted  $\ell^2$  estimates (called Hardy–Littlewood inequalities) as well as weighted  $\ell^\infty$  estimates for the coefficients of functions in  $\mathcal{H}^p$ . We discuss some consequences of these estimates and observe that the Hardy–Littlewood inequalities display what we will call a contractive symmetry between  $\mathcal{H}^p$  and  $\mathcal{H}^{4/p}$ . The relevance of the Hardy–Littlewood inequalities for the study of the dual spaces  $(\mathcal{H}^p)^*$  is illustrated by a result about the linear functionals generated by fractional primitives of the Riemann zeta function. We deduce general estimates of the norm of the partial sum operator  $\sum_{n=1}^\infty a_n n^{-s} \mapsto \sum_{n=1}^N a_n n^{-s}$  on  $\mathcal{H}^p$  with  $0 < p \leq 1$ , supplementing a classical result of Helson for the range  $1 < p < \infty$ . Finally, we discuss the relevance of our results for the computation of the so-called pseudomoments of the Riemann zeta function  $\zeta(s)$  (in the sense of Conrey and Gamburd). We apply our upper Hardy–Littlewood inequality to improve on an earlier asymptotic estimate when  $p \rightarrow \infty$ , but at the same time we show, using ideas from recent work of Harper, Nikeghbali, and Radziwiłł and some probabilistic estimates of Harper, that the Hardy–Littlewood estimate for  $p < 1$  fails to give the right asymptotics for the pseudomoments of  $\zeta^\alpha(s)$  for  $\alpha > 1$ .

## 1. INTRODUCTION

$H^p$  spaces of Dirichlet series, to be called  $\mathcal{H}^p$  in what follows, have been studied extensively in recent years but mostly in the Banach space case  $p \geq 1$ , with a view to the operators acting on them. In the present paper, we explore  $\mathcal{H}^p$  in the full range  $0 < p < \infty$ , which in part can be given a number theoretic motivation: The interplay between the additive and multiplicative structure of the integers is displayed in a more transparent way by the results obtained without any a priori restriction on the exponent  $p$ . As an example, we mention that the multiplicative estimates of Section 3 of this paper exhibit what we will call a contractive symmetry between  $H^p$  and  $H^{4/p}$ , which is particularly significant for the study of  $\mathcal{H}^p$ . We refer to these estimates as multiplicative because they are obtained by multiplicative iteration via the Bohr lift (see below) of estimates for  $H^p$  spaces of the unit disc. We note in passing that, surprisingly, there remain basic problems related to the contractive symmetry that are still open in the case of the unit disc.

By a basic observation of Bohr, the *multiplicative structure* of the integers allows us to view an ordinary Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

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as a function of infinitely many variables. Indeed, by the transformation  $z_j = p_j^{-s}$  (here  $p_j$  is the  $j$ th prime number) and the fundamental theorem of arithmetic, we have the Bohr correspondence,

$$(1) \quad f(s) := \sum_{n=1}^{\infty} a_n n^{-s} \longleftrightarrow \mathcal{B}f(z) := \sum_{n=1}^{\infty} a_n z^{\kappa(n)},$$

where we use multi-index notation and  $\kappa(n) = (\kappa_1, \dots, \kappa_j, 0, 0, \dots)$  is the multi-index such that  $n = p_1^{\kappa_1} \cdots p_j^{\kappa_j}$ . This transformation—the so-called Bohr lift—gives an isometric isomorphism between  $\mathcal{H}^p$  and the Hardy space  $H^p(\mathbb{D}^\infty)$ . We will come back to the details of this relation in the next section, where we will show that it ensures an unambiguous definition of  $\mathcal{H}^p$  in the full range  $0 < p < \infty$ . The Bohr lift is of fundamental importance in our subject, and will in particular be what we need in Section 3 and Section 4 to lift coefficient estimates in one complex variable to obtain results for  $\mathcal{H}^p$ .

The *additive structure* of the integers plays a role whenever we restrict attention to the properties of  $f(s)$  viewed as an analytic function in a half-plane or when we consider any problem for which the order of summation matters. A particularly interesting example is that of the partial sum operator

$$S_N f(s) := \sum_{n=1}^N a_n n^{-s},$$

viewed as an operator on  $\mathcal{H}^p$ . By a classical theorem of Helson [30], we know that it is uniformly bounded on  $\mathcal{H}^p$  when  $1 < p < \infty$ . In Section 5, we will give bounds that are essentially best possible in the range  $0 < p < 1$  and an improvement by a factor  $1/\log \log N$  on the previously known bounds when  $p = 1$ . We are however still far from knowing the precise asymptotics of the norm of  $S_N$  when it acts on either  $\mathcal{H}^1$  or  $\mathcal{H}^\infty$ .

We have found it interesting to relate our discussion and to apply part of our results to a number theoretic problem that deals with the *interplay* between the additive and multiplicative structure of the integers. Thus in the final Section 6 we consider the computation of the so-called pseudomoments of the Riemann zeta function  $\zeta(s)$  which were studied by Conrey and Gamburd [15] when  $p$  is an even integer. In our terminology, the pseudomoments of  $\zeta(s)$  are  $p$ th powers of the  $\mathcal{H}^p$  norms of the Dirichlet polynomials

$$Z_N(s) := \sum_{n=1}^N n^{-1/2-s}.$$

We observe that if we write

$$(2) \quad f_N(s) := \prod_{p_j \leq N} \frac{1}{1 - p_j^{-1/2-s}},$$

then  $Z_N = S_N f_N$ . Hence  $Z_N$  can be obtained by applying the partial sum operator to a Dirichlet series whose coefficients represent a completely multiplicative function. This comes as no surprise of course, but the interesting point is how to estimate the norm of  $S_N f_N$ . We have essentially two methods, one relying on the multiplicative estimates from Section 3 and another relying on an additive estimate of Helson used in Section 5. We will show that our multiplicative estimates improve on the known estimates from [8] in the range  $p > 2$ . In general, however, we know the right order of magnitude only when  $p > 1$ , there being a huge gap between the additive and multiplicative estimates in the range  $0 < p < 1$ . We are not able to remedy this situation, but we will shed light on it by showing that the  $N$ th partial sum of  $[f_N(s)]^\alpha$  for  $\alpha > 1$  has  $\mathcal{H}^p$

norm of an order of magnitude larger than what is suggested by our multiplicative estimates, provided that  $p$  is sufficiently small.

Our study of the pseudomoments of  $\zeta(s)$  and more generally  $\zeta^\alpha(s)$  highlights another important aspect of the spaces  $\mathcal{H}^p$ , namely a *probabilistic interpretation* of the Bohr correspondence and the use of probabilistic methods. Our work on pseudomoments in the range  $0 < p < 1$  is inspired by the recent paper [26] and relies crucially on some delicate probabilistic estimates due to Harper [25].

To close this introduction, we note that there are many questions about  $\mathcal{H}^p$  that are not treated or only briefly mentioned in our paper. Our selection of topics has been governed by what appear to be significant and doable problems for the whole range  $0 < p < \infty$ . We have chosen to be quite detailed in the groundwork in Section 2, dealing with the definition of  $\mathcal{H}^p$ , because the infinite-dimensional situation and the non-convexity of the  $L^p$  quasi-norms for  $0 < p < 1$  require some extra care. In that section, we also summarize briefly some known facts and easy consequences, such as for instance how some results for  $\mathcal{H}^2$  can be transferred to  $\mathcal{H}^p$  when either  $p = 2k$  or  $p = 1/(2k)$  for  $k = 2, 3, \dots$ . In Section 3, which deals with upper and lower weighted  $\ell^2$  estimates for the coefficients of functions in  $\mathcal{H}^p$ , we will record some functional analytic consequences concerning respectively duality and local embeddings of  $\mathcal{H}^p$  into appropriate Bergman spaces when  $0 < p < 2$ . For further information about known results and open problems, we refer to the monograph [37] and the recent papers [12, 43].

**Notation.** We will use the notation  $f(x) \ll g(x)$  if there is some constant  $C > 0$  such that  $|f(x)| \leq C|g(x)|$  for all (appropriate)  $x$ . If we have both  $f(x) \ll g(x)$  and  $g(x) \ll f(x)$ , we will write  $f(x) \asymp g(x)$ . If

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1,$$

then we write  $f(x) \sim g(x)$ . As above, the increasing sequence of prime numbers will be denoted by  $(p_j)_{j \geq 1}$ , and the subscript will sometimes be dropped when there can be no confusion. The number of prime factors in  $n$  will be denoted by  $\Omega(n)$  (counting multiplicities). We will also use the standard notations  $\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \leq x\}$  and  $\lceil x \rceil = \min\{n \in \mathbb{N} : n \geq x\}$ .

## 2. DEFINITIONS AND BASIC PROPERTIES OF THE HARDY SPACES $\mathcal{H}^p$ AND $H^p(\mathbb{D}^\infty)$

**2.1. Definition of  $H^p(\mathbb{D}^\infty)$ .** We use the standard notation  $\mathbb{T} := \{z : |z| = 1\}$  for the unit circle which is the boundary of the unit disc  $\mathbb{D} := \{z : |z| < 1\}$  in the complex plane, and we equip  $\mathbb{T}$  with normalized one-dimensional Lebesgue measure  $\mu$  so that  $\mu(\mathbb{T}) = 1$ . We write  $\mu_d := \mu \times \dots \times \mu$  for the product of  $d$  copies of  $\mu$ , where  $d$  may belong to  $\mathbb{N} \cup \{\infty\}$ .

We begin by recalling that for every  $p > 0$ , the classical Hardy space  $H^p(\mathbb{D})$  (also denoted by  $H^p(\mathbb{T})$ ) consists of analytic functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that

$$\|f\|_{H^p(\mathbb{D})}^p := \sup_{0 < r < 1} \int_{\mathbb{T}} |f(rz)|^p d\mu(z) < \infty.$$

This is a Banach space (quasi-Banach in case  $0 < p < 1$ ), and polynomials are dense in  $H^p(\mathbb{D})$ , so it could as well be defined as the closure of all polynomials in the above norm (or quasi-norm). We refer to [19] or the first chapters of [21] for the definition and basic properties of the Hardy spaces on  $\mathbb{D}$ .

For the finite dimensional polydisc  $\mathbb{D}^d$  with  $d \geq 2$ , the definition of Hardy spaces can be made in a similar manner: For every  $p > 0$ , a function  $f : \mathbb{D}^d \rightarrow \mathbb{C}$  belongs to  $H^p(\mathbb{D}^d)$  when it is analytic

separately with respect to each of the variables  $z_1, \dots, z_d$  and

$$\|f\|_{H^p(\mathbb{D}^d)}^p := \sup_{r < 1} \int_{\mathbb{T}^d} |f(rz)|^p d\mu_d(z) < \infty.$$

The standard source for these spaces is Rudin's monograph [41]. As in the one-dimensional case, for almost every  $z$  in  $\mathbb{T}^d$ , the radial boundary limit

$$f^*(z) := \lim_{r \rightarrow 1^-} f(rz)$$

exists, and we may write

$$(3) \quad \|f\|_{H^p(\mathbb{D}^d)}^p = \int_{\mathbb{T}^d} |f^*(z)|^p d\mu_d(z).$$

This means that  $H^p(\mathbb{D}^d)$  is a subspace of  $L^p(\mathbb{T}^d, \mu_d)$ . Moreover, again as in the one-dimensional case, for every  $f$  in  $H^p(\mathbb{D}^d)$ , we have that

$$(4) \quad \lim_{r \rightarrow 1^-} \|f - f_r\|_{H^p(\mathbb{D}^d)} = 0,$$

where  $f_r(z) := f(rz)$ . This implies that the polynomials are dense in  $H^p(\mathbb{D}^d)$ , so that the space could equally well be defined as the closure of all polynomials with respect to the norm on the boundary given by (3).

Both (3) and (4) are most conveniently obtained by applying the  $L^p$ -boundedness of the radial maximal function on  $H^p(\mathbb{D}^d)$  for all  $p > 0$ , a result which can be obtained by considering a dummy variable  $w$  in  $\mathbb{D}$  and checking first that, given  $f$  in  $H^p(\mathbb{D}^d)$ , the function

$$w \mapsto f(wz_1, \dots, wz_d)$$

lies in  $H^p(\mathbb{D}^d)$  for almost every  $(z_1, \dots, z_d) \in \mathbb{T}^d$ . By Fubini's theorem, the boundedness of the maximal function then reduces to the classical one-dimensional estimate.

In order to define  $H^p(\mathbb{D}^\infty)$ , some extra care is needed because functions in  $H^p(\mathbb{D}^\infty)$  will in general not be well defined in the whole set  $\mathbb{D}^\infty$ . To keep things simple, we henceforth consider the set  $\mathbb{D}_{\text{fin}}^\infty$  which consists of elements  $z = (z_j)_{j \geq 1} \in \mathbb{D}^\infty$  such that  $z_j \neq 0$  only for finitely many  $k$ . A function  $f : \mathbb{D}_{\text{fin}}^\infty \rightarrow \mathbb{C}$  is analytic if it is analytic at every point  $z$  in  $\mathbb{D}_{\text{fin}}^\infty$  separately with respect to each variable. Obviously any analytic  $f : \mathbb{D}_{\text{fin}}^\infty \rightarrow \mathbb{C}$  can be written by a convergent Taylor series

$$f(z) = \sum_{\kappa \in \mathbb{N}_{\text{fin}}^\infty} c_\kappa z^\kappa, \quad z \in \mathbb{D}_{\text{fin}}^\infty,$$

and the coefficients  $c_\kappa$  determine  $f$  uniquely. The truncation  $A_m f$  of  $f$  onto the first  $m$  variables  $A_m f$  (called “der *m*te Abschnitt” by Bohr) is defined as

$$A_m f(z_1, z_2, \dots) = f(z_1, \dots, z_m, 0, 0, \dots)$$

for every  $z$  in  $\mathbb{D}_{\text{fin}}^\infty$ . By applying the fundamental estimate  $|g(0)| \leq \|g\|_{H^p(\mathbb{D}^d)}$ , obtained by iterating the case  $d = 1$ , we deduce that

$$(5) \quad \|A_m f\|_{H^p(\mathbb{D}^m)} \leq \|A_{m'} f\|_{H^p(\mathbb{D}^{m'})}$$

whenever  $m' \geq m$ .

**Definition.** Let  $p > 0$ . The space  $H^p(\mathbb{D}^\infty)$  is the space of analytic functions on  $\mathbb{D}_{\text{fin}}^\infty$  obtained by taking the closure of all polynomials in the norm (quasi-norm for  $0 < p < 1$ )

$$\|f\|_{H^p(\mathbb{D}^\infty)}^p := \int_{\mathbb{T}^\infty} |f(z)|^p d\mu_\infty(z).$$

Fix a compact set  $K$  in  $\mathbb{D}^d$  and embed it as the subset  $\tilde{K}$  of  $\mathbb{D}^\infty$  so that

$$\tilde{K} := \{z = (z_1, \dots, z_d, 0, 0, \dots) \in \mathbb{D}^\infty : (z_1, \dots, z_d) \subset K\}.$$

For all polynomials  $g$  we clearly have  $\sup_{z \in \tilde{K}} |g(z)| \leq C_K \|g\|_{H^p(\mathbb{D}^\infty)}$ . It follows that any limit of polynomials is analytic on  $\mathbb{D}_{\text{fin}}^\infty$ , whence  $H^p(\mathbb{D}^\infty)$  is well defined. This also implies that every element  $f$  in  $H^p(\mathbb{D}^\infty)$  has a well-defined Taylor series  $f(z) = \sum_\kappa c_\kappa z^\kappa$  and, in turn, this Taylor series determines  $f$  uniquely. Namely, by recalling (5), we have that  $A_m f$  is in  $H^p(\mathbb{D}^m)$  for every  $m \geq 1$  and the  $A_m f$  are certainly determined by the Taylor series. Finally, by polynomial approximation, it follows that

$$\lim_{m \rightarrow \infty} \|f - A_m f\|_{H^p(\mathbb{D}^\infty)} = 0.$$

Obviously, if a function  $f$  in  $H^p(\mathbb{D}^\infty)$  depends only on the variables  $z_1, \dots, z_d$ , then we have  $\|f\|_{H^p(\mathbb{D}^\infty)} = \|f\|_{H^p(\mathbb{D}^d)}$ .

Cole and Gamelin [15] established an optimal estimate for point evaluations on  $H^p(\mathbb{D}^\infty)$  by showing that

$$(6) \quad |f(z)| \leq \left( \prod_{j=1}^{\infty} \frac{1}{1 - |z_j|^2} \right)^{1/p} \|f\|_{H^p(\mathbb{D}^\infty)}.$$

Thus the elements in the Hardy spaces continue analytically to the set  $\mathbb{D}^\infty \cap \ell^2$ .

If  $f$  is an integrable function (or a Borel measure) on  $\mathbb{T}^\infty$ , then we denote its Fourier coefficients by

$$\hat{f}(\kappa) := \int_{\mathbb{T}^\infty} f(z) \bar{z}^\kappa d\mu_\infty(z)$$

for multi-indices  $\kappa$  in  $\mathbb{Z}_{\text{fin}}^\infty$ . When  $p \geq 1$ , it follows directly from the definition of  $H^p(\mathbb{D}^\infty)$  that it can be identified as the analytic subspace of  $L^p(\mathbb{T}^\infty)$ , consisting of the elements in  $L^p(\mathbb{T}^\infty)$  whose non-zero Fourier coefficients lie in the positive cone  $\mathbb{N}_{\text{fin}}^\infty$  (called the “narrow cone” by Helson [31]).

The following result verifies that, alternatively,  $H^p(\mathbb{D}^\infty)$  may be defined in terms of the uniform boundedness of the  $L^p$ -norm of the sequence  $A_m f$  for  $m \geq 1$ , and the functions  $A_m f$  approximate  $f$  in the norm of  $H^p(\mathbb{D}^\infty)$ .

**Theorem 2.1.** *Suppose that  $0 < p < \infty$  and that  $f$  is a formal infinite dimensional Taylor series. Then  $f$  is in  $H^p(\mathbb{D}^\infty)$  if and only if*

$$(7) \quad \sup_{m \geq 1} \|A_m f\|_{H^p(\mathbb{D}^m)} < \infty.$$

Moreover, for every  $f$  in  $H^p(\mathbb{D}^\infty)$ , it holds that  $\|A_m f - f\|_{H^p(\mathbb{D}^\infty)} \rightarrow 0$  as  $m \rightarrow \infty$ .

*Proof for the case  $p \geq 1$ .* When  $p > 1$ , the statements follow from the fact that  $(A_m f)_{m \geq 1}$  is obviously an  $L^p$ -martingale sequence with respect to the natural sigma-algebras. It follows in particular that there is an  $L^p$  limit function (still denoted by  $f$ ) of the sequence  $A_m f$  on the distinguished boundary  $\mathbb{T}^\infty$ , which has the right Fourier series, and the density of polynomials follows immediately from the finite-dimensional approximation. In the case  $p = 1$ , this fact is stated in [1, Cor. 3], and is derived as consequence of the infinite-dimensional version of the brothers Riesz theorem on the absolute continuity of analytic measures, due to Helson and Lowdenslager [32] (a simpler proof of the result from [32] is also contained in [1]). The approximation property of the  $A_m f$  then follows easily.  $\square$

The case  $0 < p < 1$  requires a new argument and will be presented in the next subsection.

**2.2. Proof of Theorem 2.1 for  $0 < p < 1$ .** Our aim is to prove Lemma 2.3 below, from which the claim will follow easily. In an effort to make the computations of this section more readable, we temporarily adopt the convention that  $\|f\|_{L^p(\mathbb{T}^d)} = \|f\|_p$ , where it should be clear from the context what  $d$  is. We start with the following basic estimate.

**Lemma 2.2.** *Let  $0 < p < 1$ . There is a constant  $C_p < \infty$  such that all (analytic) polynomials  $f$  on  $\mathbb{T}$  satisfy the inequality*

$$(8) \quad \|f - f(0)\|_p^p \leq C_p \left( \|f\|_p^p - |f(0)|^p + |f(0)|^{p-p^2/2} (\|f\|_p^p - |f(0)|^p)^{p/2} \right).$$

*Proof.* In this proof, we use repeatedly the elementary inequality  $|a + b|^p \leq |a|^p + |b|^p$ , which is our replacement for the triangle inequality. We see in particular, by this inequality and the presence of the term  $\|f\|_p^p - |f(0)|^p$  inside the brackets on the right-hand side, that (8) is trivial if, say,  $\|f\|_p^p \geq (3/2)|f(0)|^2$ . We may therefore disregard this case and assume that  $f$  satisfies  $f(0) = 1$  and  $\|f\|_p^p = 1 + \varepsilon$  with  $\varepsilon < 1/2$ . Our aim is to show that, under this assumption,

$$(9) \quad \|f - 1\|_p^p \leq C_p \varepsilon^{p/2}.$$

We begin by writing  $f = UI$ , where  $U$  is an outer function and  $I$  is an inner function, such that  $U(0) > 0$ . By subharmonicity of  $|U|^p$ , we have  $1 \leq |U(0)| \leq (1 + \varepsilon)^{1/p} \leq 1 + c_p \varepsilon$ . This means that  $I(0) \geq (1 + c_p \varepsilon)^{-1} \geq 1 - c_p \varepsilon$ . We write  $f - 1 = (U - 1)I + I - 1$  and obtain consequently that

$$(10) \quad \|f - 1\|_p^p \leq \|U - 1\|_p^p + \|I - 1\|_p^p.$$

In order to prove (9), it is therefore enough to show that each of the two summands on the right-hand side of (10) is bounded by a constant times  $\varepsilon^{p/2}$ .

We begin with the second summand on the right-hand side of (10) for which we claim that

$$(11) \quad \|I - 1\|_p^p \leq C'_p \varepsilon^{p/2}$$

holds for some constant  $C'_p$ . We write  $I = u + iv$ , where  $u$  and  $v$  are respectively the real and imaginary part of  $I$ . Since  $1 - u \geq 0$ , we see that

$$(12) \quad \|1 - u\|_1 = \int_{\mathbb{T}} (1 - u(z)) dm(z) = 1 - I(0) \leq c_p \varepsilon.$$

Using Hölder's inequality, we therefore find that

$$(13) \quad \|1 - u\|_p^p \leq c_p^p \varepsilon^p.$$

In view of (12) and using that  $|I| = 1$  and  $(1 - u^2) \leq 2(1 - u)$ , we also get that

$$\|v\|_p^p \leq \|v\|_2^p = \|1 - u^2\|_1^{p/2} \leq (2\|1 - u\|_1)^{p/2} \leq (2c_p)^{p/2} \varepsilon^{p/2}.$$

Combining this inequality with (13), we get the desired bound (11).

We turn next to the first summand on the right-hand side of (10) and the claim that

$$(14) \quad \|U - 1\|_p^p \leq C''_p \varepsilon^{p/2}$$

holds for some constant  $C''_p$ . By orthogonality, we find that

$$\|U^{p/2} - U(0)^{p/2}\|_2^2 \leq \varepsilon$$

and hence

$$(15) \quad \|U^{p/2} - 1\|_2 \leq \|U^{p/2} - U(0)^{p/2}\|_2 + (U(0)^{p/2} - 1)^{1/2} \leq 2\varepsilon^{1/2}.$$



Since  $|U^{p/2} - 1| \geq ||U|^{p/2} - 1| \geq (p/2) \log_+ |U|$  and  $U(0) \geq 1$ , this implies that

$$(16) \quad \|\log |U|\|_1 = 2\|\log_+ |U|\|_1 - \log |U(0)| \leq 8p^{-1}\varepsilon^{1/2}.$$

It follows that

$$m(\{z : |\log |U(z)|| \geq \lambda\}) \leq 8(p\lambda)^{-1}\varepsilon^{1/2} \quad \text{and} \quad m(\{z : |\arg U(z)| \geq \lambda\}) \leq C\lambda^{-1}\varepsilon^{1/2},$$

where the latter inequality is the classical weak-type  $L^1$  estimate for the conjugation operator. We now split  $\mathbb{T}$  into three sets

$$\begin{aligned} E_1 &:= \{z : |U(z)| > 3/2\} \cup \{z : |U(z)| < 1/2\}, \\ E_2 &:= \{z : 1/2 \leq |U(z)| \leq 3/2, |\arg U(z)| \geq \pi/4\}, \\ E_3 &:= \mathbb{T} \setminus (E_1 \cup E_2). \end{aligned}$$

It is immediate from (15) that

$$\|\chi_{E_1}(U - 1)\|_p^p \ll \varepsilon.$$

Since  $m(E_2) \leq C\varepsilon^{1/2}$ , we have trivially that

$$\|\chi_{E_2}(U - 1)\|_p^p \leq C(5/2)^p \varepsilon^{1/2}.$$

Finally, on  $E_3$ , we have that  $|U^{p/2} - 1| \simeq |U - 1|$ , and so it follows from (15) and Hölder's inequality that

$$\|\chi_{E_3}(U - 1)\|_p^p \ll \varepsilon^{p/2}.$$

Now the desired inequality (14) follows by combining the latter three estimates.  $\square$

One may notice that that in the last step of the proof above we could have used (16) and the fact that the conjugation operator is bounded from  $L^1$  to  $L^p$ . It seems that the exponent  $p/2$  is the best we can get. It is also curious to note that with  $p = 2/k$  and  $k \geq 2$  an integer, one could avoid the use of the weak-type estimate for  $\arg U$  and get a very slick argument by simply observing that if  $g = U^{p/2}$  and  $\omega_1, \dots, \omega_k$  are the  $k$ th roots of unity, then by Hölder's inequality,

$$\|U - 1\|_p \leq \prod_{j=1}^k \|g - \omega_j\|_2,$$

and on the right hand side one  $L^2$ -norm is estimated by  $\varepsilon^{1/2}$  and the others by a constant since we are assuming  $\varepsilon \leq 1/2$ . Again one could raise the question if one can interpolate to get all exponents.

**Lemma 2.3.** *Suppose that  $0 < p < 1$ . If  $g$  is a polynomial on  $\mathbb{T}^\infty$ , then*

$$\|A_{m+k}g - A_mg\|_p^p \leq C_p \left( \|A_{m+k}g\|_p^p - \|A_mg\|_p^p + \|A_mg\|_p^{p-p^2/2} (\|A_{m+k}g\|_p^p - \|A_mg\|_p^p)^{p/2} \right)$$

*holds for arbitrary positive integers  $m$  and  $k$ , where  $C_p$  is as in Lemma 8.*

*Proof.* We set  $h := A_{m+k}g$  and view  $h$  as a function on  $\mathbb{T}^m \times \mathbb{T}^k$  so that  $A_mg(w, w') = h(w, 0)$ . Now fix arbitrary points  $w$  in  $\mathbb{T}^m$  and  $w'$  in  $\mathbb{T}^k$ . We apply the preceding lemma to the function

$$f(z) := h(w, zw'),$$

which is an analytic function on  $\mathbb{D}$ . This yields

$$\begin{aligned} \int_{\mathbb{T}} |h(w, zw') - h(w, 0)|^p d\mu(z) &\leq C_p \left( \int_{\mathbb{T}} |h(w, zw')|^p d\mu(z) - |h(w, 0)|^p \right. \\ &\quad \left. + |h(w, 0)|^{p-p^2/2} \left( \int_{\mathbb{T}} |h(w, zw')|^p d\mu(z) - |h(w, 0)|^p \right)^{p/2} \right). \end{aligned}$$

The claim follows by integrating both sides with respect to  $(w, w')$  over  $\mathbb{T}^{m+k}$  and applying Hölder's inequality to the last term on the right-hand side.  $\square$

*Proof of Theorem 2.1 for  $0 < p < 1$ .* If  $f$  is in  $H^p(\mathbb{D}^\infty)$ , then clearly (7) holds. To prove the reverse implication, we start from a formal Taylor series  $f$  for which (7) holds. Then by assumption  $A_m f$  is in  $H^p(\mathbb{D}^\infty)$ , and we have that  $A_m(A_{m'} f) = A_m f$  whenever  $m' \geq m \geq 1$ . Therefore the quasi-norms  $\|A_m f\|_{H^p(\mathbb{D}^\infty)}$  constitute an increasing sequence, and hence (7) implies that

$$\lim_{m \rightarrow \infty} \sup_{k \geq 1} (\|A_{m+k} f\|_{H^p(\mathbb{D}^\infty)} - \|A_m f\|_{H^p(\mathbb{D}^\infty)}) = 0.$$

By Lemma 2.3, we therefore find that  $(A_m f)_{m \geq 1}$  is a Cauchy sequence in  $H^p(\mathbb{D}^\infty)$ , whence  $f = \lim_{m \rightarrow \infty} A_m f$  in  $H^p(\mathbb{D}^\infty)$  since an element in  $H^p(\mathbb{D}^\infty)$  is uniquely determined by the sequence  $A_m f$ .  $\square$

**2.3. Definition of  $\mathcal{H}^p$ .** The systematic study of the Hilbert space  $\mathcal{H}^2$  began with the paper [29] which defined  $\mathcal{H}^2$  to be the collection of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

subject to the condition  $\|f\|_{\mathcal{H}^2}^2 := (\sum_{n=1}^{\infty} |a_n|^2)^{1/2} < \infty$ . The space  $\mathcal{H}^2$  consists of functions analytic in the half-plane  $\mathbb{C}_{1/2} := \{s = \sigma + it : \sigma > 1/2\}$ , since the Cauchy–Schwarz inequality shows that the above Dirichlet series converges absolutely for those values of  $s$ . Bayart [5] extended the definition to every  $p > 0$  by defining  $\mathcal{H}^p$  as the closure of all Dirichlet polynomials  $f(s) := \sum_{n=1}^N a_n n^{-s}$  under the norm (or quasi-norm when  $0 < p < 1$ )

$$(17) \quad \|f\|_{\mathcal{H}^p} := \left( \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(it)|^p dt \right)^{1/p}.$$

Computing the limit when  $p = 2$ , we see that (17) gives back the original definition of  $\mathcal{H}^2$ . However, at first sight it is not clear that the above definition of  $\mathcal{H}^p$  is the right one or that it even yields spaces of convergent Dirichlet series in any right half-plane.

The clarification of these matters is provided by the Bohr lift (1). By Birkhoff's ergodic theorem (or by an elementary argument found in [44, Sec. 3]), we obtain the identity

$$(18) \quad \|f\|_{\mathcal{H}^p} = \|\mathcal{B}f\|_{H^p(\mathbb{D}^\infty)} := \left( \int_{\mathbb{T}^\infty} |\mathcal{B}f(z)|^p d\mu_\infty(z) \right)^{1/p}.$$

Since the Hardy spaces on the infinite dimensional torus  $H^p(\mathbb{D}^\infty)$  may be defined as the closure of analytic polynomials in the  $L^p$ -norm on  $\mathbb{T}^\infty$ , it follows that the Bohr correspondence gives an isomorphism between the spaces  $H^p(\mathbb{D}^\infty)$  and  $\mathcal{H}^p$ . This linear isomorphism is both isometric and multiplicative, and this results in a fruitful interplay: Many questions in the theory of the spaces  $\mathcal{H}^p$  can be better treated by considering the isomorphic space  $H^p(\mathbb{D}^\infty)$ , and vice versa. An important example is the Cole–Gamelin estimate (6) which immediately implies that for



every  $p > 0$  the space  $\mathcal{H}^p$  consists of analytic functions in the half-plane  $\mathbb{C}_{1/2}$ . In fact, we infer from (6) that

$$|f(\sigma + it)|^p \leq \zeta(2\sigma) \|f\|_{\mathcal{H}^p}^p$$

holds whenever  $\sigma > 1/2$ , where  $\zeta(s)$  is the Riemann zeta function. Moreover, since the coefficients of a convergent Dirichlet series are unique, functions in  $\mathcal{H}^p$  are completely determined by their restrictions to the half-plane  $\mathbb{C}_{1/2}$ . This means in particular that  $\mathcal{H}^p$  can be thought of as a space of analytic functions in this half-plane.

To complete the picture, we mention that  $\mathcal{H}^\infty$  is defined as the space of Dirichlet series  $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  that represent bounded analytic functions in the half-plane  $\sigma > 0$ . We endow  $\mathcal{H}^\infty$  with the norm

$$\|f\|_{\mathcal{H}^\infty} := \sup_{\sigma > 0} |f(s)|, \quad s = \sigma + it,$$

and then the Bohr lift allows us to associate  $\mathcal{H}^\infty$  with  $H^\infty(\mathbb{D}^\infty)$ . We refer to [37] for this fact and further details about the interesting and rich function theory of  $\mathcal{H}^\infty$ .

**2.4. A probabilistic interpretation of the Bohr lift.** It is frequently fruitful to think of the product measure  $\mu_\infty$  on  $\mathbb{T}^\infty$  as a probability measure and the infinitely many variables  $z_k$  as independent identically distributed (i.i.d.) random variables. From the viewpoint of the Bohr correspondence, we then associate with the sequence of primes  $(p_j)_{j \geq 1}$  a sequence of independent Steinhaus variables  $z(p_j)$ , which are random variables equidistributed on  $\mathbb{T}$ . This sequence defines a random multiplicative function  $z(n)$  on the positive integers  $\mathbb{N}$  by the rule

$$z(n) = (z(p_j))^{\kappa(n)},$$

where we again use multi-index notation. Functions in  $\mathcal{H}^p$  can then, via the Bohr lift, be thought of as linear combinations of these random multiplicative functions. Indeed, we may write the Bohr lift as

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \longleftrightarrow F((z(p_j))) = \sum_{n=1}^{\infty} a_n z(n)$$

and hence express  $\|f\|_{\mathcal{H}^p}^p$  as the  $p$ th moment of  $|F|$ :

$$\|f\|_{\mathcal{H}^p}^p = \mathbb{E}|F|^p.$$

In the final section of this paper, we will make crucial use of this alternate viewpoint, and we then find it natural to switch to this probabilistic terminology.

**2.5. Summary of known results.** The function theory of the two distinguished spaces  $\mathcal{H}^2$  and  $\mathcal{H}^\infty$  is by now quite well developed; we refer again to [37, 43] for details. The results for the range  $1 \leq p < \infty$ ,  $p \neq 2$ , are less complete. In this section, we mention briefly some key results that extend to the whole range  $0 < p < \infty$ , as well as some familiar difficulties that arise in our attempts to make such extensions.

We begin with the theorem on multipliers that was first established in [29] for  $p = 2$  and extended to the range  $1 \leq p < \infty$  in [5]. We recall that a multiplier  $m$  for  $\mathcal{H}^p$  is a function such that the operator  $f \mapsto mf$  is bounded on  $\mathcal{H}^p$ , and the multiplier norm is the norm of this operator. The theorem on multipliers asserts that the space of multipliers for  $\mathcal{H}^p$  is equal to  $\mathcal{H}^\infty$ , and this remains true for  $0 < p < 1$ , by exactly the same proof as in [5]. Another result that carries over without any change, is the Littlewood–Paley formula of [7, Sec. 5]. The latter result was already used in [12].

For some results, only a partial extension from the case  $p = 2$  is known to hold. A well known example is whether the  $L^p$  integral of a Dirichlet polynomial  $f(s) = \sum_{n=1}^N a_n n^{-s}$  over any segment of fixed length on the vertical line  $\operatorname{Re} s = 1/2$  is bounded by a universal constant times  $\|f\|_{\mathcal{H}^p}^p$ . This is known to hold for  $p = 2$  and thus trivially for  $p = 2k$  for  $k$  a positive integer. As shown in [36], this embedding holds if and only if the following is true: The boundedly supported Carleson measures for  $\mathcal{H}^p$  satisfy the classical Carleson condition in  $\mathbb{C}_{1/2}$ .

There is an interesting counterpart for  $p < 2$  to the trivial embedding for  $p = 2k$  and  $k$  a positive integer  $> 1$ . This is the following statement about interpolating sequences. If  $S = (s_j)$  is a bounded interpolating sequence in  $\mathbb{C}_{1/2}$ , then we can solve the interpolation problem  $f(s_j) = a_j$  in  $\mathcal{H}^p$  when

$$\sum_j |a_j|^p (2\sigma_j - 1) < \infty$$

and  $p = 2/k$  for  $k$  a positive integer. Indeed, choose any  $k$ th root  $a_j^{1/k}$  and solve  $g(s_j) = a_j^{1/k}$  in  $\mathcal{H}^2$ . Then  $f = g^k$  solves our problem in  $\mathcal{H}^p$ . We do not know if this result extends to any  $p$  which is not of the form  $p = 2/k$ . Comparing the two trivial cases, we observe that there is an interesting “symmetry” between the embedding problem for  $\mathcal{H}^p$  and the interpolation problem for  $\mathcal{H}^{4/p}$ . A similar phenomenon will be explored in the next section.

Before turning to the next two sections which will deal with respectively weighted  $\ell^2$  and  $\ell^\infty$  bounds for the coefficients, we would like to point out that there are certainly other interesting problems of a similar kind. An interesting example is whether the  $\ell^1$  estimate

$$\sum_{n=2}^{\infty} \frac{|a_n|}{\sqrt{n} \log n} \leq C \|f\|_{\mathcal{H}^1}$$

holds when  $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ . We refer to [13] for background on this problem and again to [43] for a survey of other open problems.

### 3. COEFFICIENT ESTIMATES: WEIGHTED $\ell^2$ BOUNDS

**3.1. Contractive Hardy–Littlewood inequalities in the unit disc.** We begin with some estimates of the  $H^p(\mathbb{D})$  norms (or quasi-norms when  $0 < p < 1$ ) in terms of weighted  $\ell^2$  norms of the coefficient sequence. Such inequalities were first studied systematically by Hardy and Littlewood.

For  $\alpha > 1$ , the weighted Bergman space  $A_\alpha^p(\mathbb{D})$  is the space of analytic functions on  $\mathbb{D}$  for which

$$\|f\|_{A_\alpha^p(\mathbb{D})} := \left( \int_{\mathbb{D}} |f(z)|^p (\alpha - 1) (1 - |z|^2)^{\alpha-2} \frac{dm(z)}{\pi} \right)^{\frac{1}{p}} < \infty,$$

where  $m$  denotes Lebesgue area measure on  $\mathbb{C}$ . We set

$$dm_\alpha(z) := (\alpha - 1) (1 - |z|^2)^{\alpha-2} \frac{dm(z)}{\pi}.$$

The Hardy space  $H^p(\mathbb{D})$  is the limit of the weighted Bergman spaces  $A_\alpha^p(\mathbb{D})$  as  $\alpha \rightarrow 1^+$ , in the sense that

$$\|f\|_{H^p(\mathbb{D})} = \lim_{\alpha \rightarrow 1^+} \|f\|_{A_\alpha^p(\mathbb{D})}.$$

We will therefore find it convenient to write  $H^p(\mathbb{D}) = A_1^p(\mathbb{D})$  in some formulas, such as in (22) below. For  $\alpha \geq 1$  and a non-negative integer  $j$ , we define

$$(19) \quad c_\alpha(j) := \binom{j + \alpha - 1}{j} = \prod_{l=1}^{\alpha-1} \frac{(j+l)}{l}.$$

Notice that  $c_1(j) = 1$  for every  $j$ . Identifying  $c_\alpha(j)$  as the coefficients of the binomial series  $(1-z)^{-\alpha}$ , we find that

$$(20) \quad c_{\alpha k}(j) = \sum_{j_1 + j_2 + \dots + j_k = j} c_\alpha(j_1) c_\alpha(j_2) \cdots c_\alpha(j_k).$$

In particular, if  $\alpha$  is a positive integer, then  $c_\alpha(j)$  is the number of ways to write  $j$  as a sum of  $\alpha$  non-negative integers. We will also require the simple estimate

$$(21) \quad c_\alpha(j+k) \leq c_\alpha(j) c_\alpha(k)$$

which can be deduced by comparing factor by factor in the product (19). A computation gives that if  $f(z) = \sum_{j \geq 0} a_j z^j$ , then

$$(22) \quad \|f\|_{A_\alpha^2(\mathbb{D})} = \left( \sum_{j=0}^{\infty} \frac{|a_j|^2}{c_\alpha(j)} \right)^{\frac{1}{2}}.$$

The following inequality is due to Burbea [14, Cor. 3.4], but we include a short proof in the special case we require, based on (20).

**Lemma 3.1.** *Suppose that  $f$  is in  $H^2(\mathbb{D})$ , and let  $k$  be an integer  $\geq 2$ . Then*

$$\|f\|_{A_k^{2k}(\mathbb{D})} = \left( \int_{\mathbb{D}} |f(z)|^{2k} dm_k(z) \right)^{\frac{1}{2k}} \leq \|f\|_{H^2(\mathbb{D})}.$$

*Proof.* Suppose that  $f(z) = \sum_{j=0}^{\infty} a_j z^j$ . We write  $|f|^{2k} = |f^k|^2$  and use (22), followed by the Cauchy–Schwarz inequality with (20), to get

$$\begin{aligned} \|f\|_{A_k^{2k}(\mathbb{D})}^{2k} &= \sum_{j=0}^{\infty} \frac{1}{c_k(j)} \left| \sum_{j_1 + \dots + j_k = j} a_{j_1} \cdots a_{j_k} \right|^2 \\ &\leq \sum_{j=0}^{\infty} \sum_{j_1 + \dots + j_k = j} |a_{j_1}|^2 \cdots |a_{j_k}|^2 = \left( \sum_{j=0}^{\infty} |a_j|^2 \right)^{2k} = \|f\|_{H^2(\mathbb{D})}^{2k}. \end{aligned} \quad \square$$

It is possible to use Lemma 3.1 and Riesz–Thorin interpolation to prove that

$$(23) \quad \|f\|_{A_k^{2k}(\mathbb{D})} = \left( \int_{\mathbb{D}} |f(z)|^{2k} dm_k(z) \right)^{\frac{1}{2k}} \leq C_k \|f\|_{H^2(\mathbb{D})}$$

holds for every real number  $k > 1$ , but the interpolation process gives a constant  $C_k > 1$  when  $k$  is not an integer. Numerical evidence has been supplied elsewhere [11] for the conjecture that in fact  $C_k = 1$  for all  $k > 1$ . So far, we have not been able to prove this extension of Lemma 3.1. As a remedy for this situation, we will establish a weaker version for all  $k$  which will be satisfactory for the number theoretic applications to be discussed later. For this, we need the following remarkable contractive estimate of Weissler [53, Cor. 2.1] for the dilations

$$f_r(z) := f(rz), \quad r > 0,$$

of functions  $f$  in  $H^p(\mathbb{D})$ .

**Lemma 3.2.** *Let  $0 < p < q < \infty$ . The contractive estimate*

$$\|f_r\|_{H^q(\mathbb{D})} \leq \|f\|_{H^p(\mathbb{D})}$$

*holds for every  $f$  in  $H^p(\mathbb{D})$  if and only if  $r \leq \sqrt{p/q}$ .*

We are now ready to state and prove the main result of this section. To this end, we set

$$(24) \quad \varphi_\alpha(j) := c_{\lfloor \alpha \rfloor}(j) \left( \frac{\alpha}{\lfloor \alpha \rfloor} \right)^j, \quad \alpha \geq 1.$$

**Theorem 3.3.** *Suppose that  $0 < p < \infty$  and that  $f(z) = \sum_{j=0}^{\infty} a_j z^j$  is in  $H^p(\mathbb{D})$ . Then*

$$(25) \quad \|f\|_{H^p(\mathbb{D})} \leq \left( \sum_{j=0}^{\infty} |a_j|^2 \varphi_{p/2}(j) \right)^{\frac{1}{2}}, \quad p \geq 2,$$

$$(26) \quad \left( \sum_{j=0}^{\infty} \frac{|a_j|^2}{\varphi_{2/p}(j)} \right)^{\frac{1}{2}} \leq \|f\|_{H^p(\mathbb{D})}, \quad p \leq 2.$$

*Here the respective parameters  $\alpha = p/2$  and  $\alpha = 2/p$  are optimal for contractivity.*

*Proof.* We begin with (25). We will use Lemma 3.2 in reverse with exponents  $\lfloor p/2 \rfloor = k$  and  $p/2$ , so we choose  $r^2 = \lfloor p/2 \rfloor / (p/2)$  and assume that  $f$  is a polynomial. Hence

$$\|f\|_{H^p(\mathbb{D})} = \|f^2\|_{H^{p/2}(\mathbb{D})}^{1/2} \leq \|f_{1/r}^2\|_{H^k(\mathbb{D})}^{1/2} = \|f_{1/r}^k\|_{H^2(\mathbb{D})}^{\frac{1}{k}}.$$

The right-hand side can be computed at the level of coefficients. We use the Cauchy–Schwarz inequality with (20), and finally (21), to get

$$\begin{aligned} \|f_{1/r}^k\|_{H^2(\mathbb{D})}^{\frac{1}{k}} &= \left( \sum_{j=0}^{\infty} \left| \sum_{j_1+\dots+j_k=j} a_{j_1} r^{-j_1} \dots a_{j_k} r^{-j_k} \right|^2 \right)^{\frac{1}{2k}} \\ &\leq \left( \sum_{j=0}^{\infty} c_k(j) \sum_{j_1+\dots+j_k=j} |a_{j_1}|^2 r^{-2j_1} \dots |a_{j_k}|^2 r^{-2j_k} \right)^{\frac{1}{2k}} \\ &\leq \left( \sum_{j=0}^{\infty} \sum_{j_1+\dots+j_k=j} |a_{j_1}|^2 c_k(j_1) r^{-2j_1} \dots |a_{j_k}|^2 c_k(j_k) r^{-2j_k} \right)^{\frac{1}{2k}} \\ &= \left( \sum_{j=0}^{\infty} |a_j|^2 c_k(j) r^{-2j} \right)^{\frac{1}{2}}. \end{aligned}$$

This completes the proof of (25), since  $k = \lfloor p/2 \rfloor$  and  $r^2 = \lfloor p/2 \rfloor / (p/2)$ , so

$$c_k(j) r^{-2j} = c_{\lfloor p/2 \rfloor}(j) \left( \frac{p/2}{\lfloor p/2 \rfloor} \right)^j = \varphi_{p/2}(j).$$

To prove (26), we first assume that  $f(z) \neq 0$  for every  $z$  in  $\mathbb{D}$  so  $f^{1/k}$  is an analytic function for  $k = \lfloor 2/p \rfloor$ . We then use Lemma 3.2 with exponents  $kp$  and 2, followed by Lemma 3.1, to get

$$(27) \quad \|f\|_{H^p(\mathbb{D})} = \|f^{1/k}\|_{H^{kp}(\mathbb{D})}^k \geq \|f_{kp/2}^{1/k}\|_{H^2(\mathbb{D})}^k \geq \|f_{kp/2}\|_{A_k^2(\mathbb{D})}.$$

If  $f \neq 0$ , we factor  $f = Bg$ , where  $B$  is a Blaschke product and  $g$  has no zeros in  $\mathbb{D}$ . Then

$$\|f\|_{H^p(\mathbb{D})} = \|g\|_{H^p(\mathbb{D})} \quad \text{and} \quad \|f_{kp/2}\|_{A_k^2(\mathbb{D})} \leq \|g_{kp/2}\|_{A_k^2(\mathbb{D})}.$$

It therefore follows that (27) is valid for every  $f$  in  $H^p(\mathbb{D})$ . The right hand side of (27) can be computed at the level of coefficients, and since  $k = \lfloor 2/p \rfloor$  we find that

$$\|f_{kp/2}\|_{A_k^2(\mathbb{D})} = \left( \sum_{j=0}^{\infty} \frac{|a_j|^2}{c_k(j)} \left( \frac{kp}{2} \right)^j \right)^{\frac{1}{2}} = \left( \sum_{j=0}^{\infty} \frac{|a_j|^2}{\varphi_{2/p}(j)} \right)^{\frac{1}{2}},$$

which completes the proof of (26). To see that  $\alpha = p/2$  and  $\alpha = 2/p$  are optimal, we recall that  $\varphi_\alpha(1) = \alpha$  and, for  $0 < \varepsilon < 1$ , compute

$$\|1 + \varepsilon z\|_{H^p(\mathbb{D})}^2 = \|(1 + \varepsilon z)^{p/2}\|_{H^2(\mathbb{D})}^{4/p} = \left( 1 + \frac{p^2}{4} \varepsilon^2 + O(\varepsilon^4) \right)^{\frac{2}{p}} = 1 + \frac{p}{2} \varepsilon^2 + O(\varepsilon^4).$$

We complete the proof by letting  $\varepsilon$  tend to 0. □

When  $1 < p \leq 2$  or  $2 \leq p < 4$ , the result of Theorem 3.3 is simply Lemma 3.2, which gives the inequalities

$$\begin{aligned} \|f\|_{H^p(\mathbb{D})} &\leq \left( \sum_{j=0}^{\infty} |a_j|^2 \left( \frac{p}{2} \right)^j \right)^{\frac{1}{2}}, & q \geq 2, \\ \left( \sum_{j=0}^{\infty} |a_j|^2 \left( \frac{2}{p} \right)^j \right)^{\frac{1}{2}} &\leq \|f\|_{H^p(\mathbb{D})}, & p \leq 2. \end{aligned}$$

The virtue of Theorem 3.3, compared to what Lemma 3.2 will give for every  $0 < p < \infty$ , is that the geometric factor  $(\alpha/\lfloor \alpha \rfloor)^j$  is always dominated by  $(2 - \delta)^j$  for some  $\delta = \delta(\alpha) > 0$ . It will become clear why this is crucial in Subsection 3.3.

**3.2. Hardy–Littlewood inequalities for  $\mathcal{H}^p$ .** We recall the definition of the Riemann zeta function,

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}}, \quad \sigma > 1.$$

Using the Euler product formula, we may define the general divisor function  $d_\alpha(n)$  by the rule

$$(28) \quad \zeta^\alpha(s) = \sum_{n=1}^{\infty} d_\alpha(n) n^{-s}, \quad \sigma > 1.$$

A basic observation is that  $d_\alpha(n)$  is a multiplicative function, which means that it is completely determined by its values at powers of the prime numbers. The Euler product formula shows that, in fact,

$$(29) \quad d_\alpha(p^j) = c_j(\alpha)$$

for every prime  $p$  and every nonnegative integer  $j$ , and in general

$$d_\alpha(n) = (c_j(\alpha))^{\kappa(n)}$$

in multi-index notation. We may thus think of  $d_\alpha(n)$  as a multiplicative extension of (29).

We will now make a multiplicative extension of Theorem 3.3, similar to the extension from (29) to  $d_\alpha(n)$ . This will be done by an iterative procedure introduced by Bayart [5] which relies

crucially on the contractivity of the estimates of Theorem 3.3. We begin by noting that the multiplicative extension of Theorem 3.3 is known when either  $p/2$  or  $2/p$  is an integer. If  $p/2$  is an integer, (our version of) the inequality in [45, Lem. 8] is

$$(30) \quad \|f\|_{\mathcal{H}^p} \leq \left( \sum_{n=1}^{\infty} |a_n|^2 d_{p/2}(n) \right)^{\frac{1}{2}}.$$

On the other hand, it was observed in [8, pp. 203–204], that (26) can be used to prove the corresponding lower inequality

$$(31) \quad \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{d_{2/p}(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{H}^p},$$

where it is required that  $2/p$  is an integer. The case  $p = 1$  in (31) is often called Helson's inequality [31]. For both (30) and (31), it is easy to see that the parameters  $\alpha = p/2$  and  $\alpha = 2/p$  are best possible — a similar statement will be proved in Theorem 3.4.

Both (30) and (31) rely on Theorem 3.3, and we do not know whether any of them extend to the case when either  $p/2 > 1$  or  $2/p > 1$ . We now turn to what we are able to prove, namely the multiplicative extension of Theorem 3.3 for general  $p$ . To this end, in accordance with (24), we introduce the multiplicative function

$$(32) \quad \Phi_{\alpha}(n) := d_{\lfloor \alpha \rfloor}(n) \left( \frac{\alpha}{\lfloor \alpha \rfloor} \right)^{\Omega(n)},$$

where  $\Omega(n)$  denotes the number of prime factors in  $n$ , counting multiplicity. We will see later that  $\Phi_{\alpha}(n)$  has the same average order as  $d_{\alpha}(n)$ , a fact that for our purposes makes it a satisfactory substitute.

The multiplicative extension of Theorem 3.3 reads as follows.

**Theorem 3.4.** *If  $f(s) = \sum_{n=1}^N a_n n^{-s}$ , then*

$$(33) \quad \|f\|_{\mathcal{H}^p} \leq \left( \sum_{n=1}^N |a_n|^2 \Phi_{p/2}(n) \right)^{\frac{1}{2}}, \quad p \geq 2,$$

$$(34) \quad \left( \sum_{n=1}^N \frac{|a_n|^2}{\Phi_{2/p}(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{H}^p}, \quad p \leq 2.$$

*The respective parameters  $\alpha = p/2$  and  $\alpha = 2/p$  are optimal.*

Observe that  $\Phi_{\alpha}(n) = d_{\alpha}(n)$  whenever  $\alpha$  is an integer, so (33) and (34) encompass (30) and (31), respectively. Note also that  $\Phi_{\alpha}(n) = d_{\alpha}(n)$  if  $n$  is square-free.

Theorem 3.4 is an improvement of results<sup>1</sup> from [8, 45]. Indeed, it is proved in [8] that (31) holds if we only consider square-free integers in the lower bound. Using the Möbius function  $\mu(n)$ , which is the multiplicative function that is 0 if  $n$  is square-free and  $-1$  at each prime number, the Hardy–Littlewood inequality of [8] can be written as

$$(35) \quad \left( \sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{d_{2/p}(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{H}^p},$$

---

<sup>1</sup>The Hardy–Littlewood inequalities in [8, 45] are stated with a weight of the form  $[d(n)]^{\beta}$ , where  $d(n) = d_2(n)$  denotes the usual divisor function. The difference between  $[d(n)]^{\beta}$  and  $d_{\alpha}(n)$  is marginal, but we have found it more natural to use  $d_{\alpha}(n)$ .



for  $p \leq 2$ . We will see in Section 6 that certain estimates obtained from (35) cannot be improved (substantially) by using (34). Nevertheless, the fact that  $\Phi_\alpha(n) > 0$  for every  $n$  allows us to extend an embedding theorem from [6] from  $1 \leq p < 2$  to  $0 < p < 2$ . This cannot be achieved using (35), since the lower bound is supported on square-free integers.

In [45], Riesz–Thorin interpolation between the integers  $p/2$  in (30) is used to prove that

$$\|f\|_{\mathcal{H}^q} \leq \left( \sum_{n=1}^{\infty} |a_n|^2 d_\alpha(n) \right)^{\frac{1}{2}},$$

where  $\alpha = \alpha(p) > p/2$  (unless  $p/2$  is an integer). Thus the average order of  $d_\alpha(n)$  is larger than that of  $d_{p/2}(n)$  and hence than that of  $\Phi_p(n)$  as well, as we will see in the next subsection.

We now turn to the proof of Theorem 3.4. It uses a technique which has become standard by now (see e.g. [5, 6, 8, 31]), and we will therefore be brief. For applications of this result, we refer to the subsequent Subsections 3.4 and 3.5 and Sections 5 and 6.

*Proof of Theorem 3.4.* By (18), we may replace  $f$  by  $\mathcal{B}f$ , and we may assume that  $\mathcal{B}f =: F$  is a polynomial. We wish to apply Theorem 3.3 iteratively to the finitely many variables  $z_j$  on which  $F$  depends. To carry out the iterative argument, we need the following integral version of Minkowski's inequality. Let  $X$  and  $Y$  be measure spaces, and let  $g$  be a measurable function on  $X \times Y$ . If  $r \geq 1$ , then

$$(36) \quad \left( \int_X \left( \int_Y |g(x, y)|^r dy \right)^{\frac{1}{r}} dx \right)^r \leq \int_Y \left( \int_X |g(x, y)|^r dx \right)^{\frac{1}{r}} dy.$$

We use the Euler product of the Riemann zeta function in (28) and recall that  $c_\alpha(j)$  are the coefficients of the binomial series  $(1 - z)^{-\alpha}$ , to conclude that  $\Phi_\alpha$  is the multiplicative function defined by

$$\Phi_\alpha(p^k) = \varphi_\alpha(k).$$

Fix  $d \geq 2$ , and define the invertible linear operator  $T_\alpha$  by

$$(37) \quad T_\alpha(z^{\kappa(n)}) := \sqrt{\Phi_\alpha(n)} z^{\kappa(n)} = \prod_{j=1}^d \sqrt{\varphi_\alpha(\kappa_j)} z^{\kappa_j}.$$

Let  $T_\alpha^{-1}$  denote the inverse operator. In view of (18), it is sufficient to prove that if  $F$  is an analytic polynomial in  $d$  variables, then

$$\begin{aligned} \|F\|_{L^p(\mathbb{T}^d)} &\leq \|T_{p/2} F\|_{L^2(\mathbb{T}^d)}, & p \geq 2, \\ \|T_{2/p}^{-1} F\|_{L^2(\mathbb{T}^d)} &\leq \|F\|_{L^p(\mathbb{T}^d)}, & p \leq 2. \end{aligned}$$

Note that Theorem 3.3 is simply the case  $d = 1$ . We find it convenient to argue by induction on  $d$ , and we will only consider the case  $p \geq 2$ . We factor the operator  $T_\alpha$  as  $T_\alpha = R_\alpha S_\alpha$ , such that  $R_\alpha$  acts on  $z_j$  for  $1 \leq j \leq d-1$  and  $S_\alpha$  acts on  $z_d$ . This is well-defined in view of (37).

The induction hypothesis becomes  $\|g\|_{L^q(\mathbb{T}^{d-1})} \leq \|R_{p/2} g\|_{L^2(\mathbb{T}^{d-1})}$ . To simplify the notation, set  $z_d = w$ . We begin by using Theorem 3.3 to the effect that

$$\begin{aligned} \|F\|_{L^p(\mathbb{T}^d)} &= \left( \int_{\mathbb{T}^{d-1}} \int_{\mathbb{T}} |F(z_1, \dots, z_{d-1}, w)|^p d\mu(w) d\mu_{d-1}(z_1, \dots, z_{d-1}) \right)^{\frac{1}{p}} \\ &\leq \left( \int_{\mathbb{T}^{d-1}} \left( \int_{\mathbb{T}} |S_{p/2} F(z_1, \dots, z_{d-1}, w)|^2 d\mu(w) \right)^{\frac{p}{2}} d\mu_{d-1}(z) \right)^{\frac{1}{p}}. \end{aligned}$$

We now apply (36) with  $X = \mathbb{T}^{d-1}$ ,  $Y = \mathbb{T}$  and  $r = p/2$ , and find that

$$\leq \left( \int_{\mathbb{T}} \left( \int_{\mathbb{T}^{d-1}} |S_{p/2} F(z_1, \dots, z_{d-1}, w)|^p d\mu_{d-1}(z) \right)^{\frac{2}{p}} d\mu(w) \right)^{\frac{1}{2}}.$$

We complete the proof by using the induction hypothesis on  $g = S_{p/2} f$ .

That the parameter  $\alpha = p/2$  is optimal, follows at once from the example considered in the proof of Theorem 3.3, applied multiplicatively.  $\square$

**3.3. The average order of  $\Phi_\alpha(n)$ .** From (28) it follows by standard techniques (see e.g. [50, Ch. II.5]) that the average order of  $d_\alpha(n)$  is given by

$$(38) \quad \frac{1}{N} \sum_{n=1}^N d_\alpha(n) = \frac{1}{\Gamma(\alpha)} (\log N)^{\alpha-1} + O((\log N)^{\alpha-2}).$$

We will now show that  $\Phi_\alpha(n)$  has the same average order, up to a bounded factor. To investigate the average order of  $\Phi_\alpha(n)$ , we consider the associated Dirichlet series and factor out a suitable power of  $\zeta(s)$  from the Euler product, to obtain

$$\mathcal{F}_\alpha(s) := \sum_{n=1}^{\infty} \Phi_\alpha(n) n^{-s} = \zeta^\alpha(s) \prod_p (1 - p^{-s})^\alpha \left( \sum_{j=0}^{\infty} \varphi_\alpha(j) p^{-js} \right).$$

For  $|z| < \lfloor \alpha \rfloor / \alpha$ , it is now convenient to set

$$(39) \quad G_\alpha(z) := (1-z)^\alpha \sum_{j=0}^{\infty} \varphi_\alpha(j) z^j = (1-z)^\alpha \left( 1 - \frac{\alpha}{\lfloor \alpha \rfloor} z \right)^{-\lfloor \alpha \rfloor}$$

so that  $\mathcal{F}_\alpha(s) = \zeta^\alpha(s) \mathcal{G}_\alpha(s)$ , where

$$\mathcal{G}_\alpha(s) := \prod_p G_\alpha(p^{-s}).$$

From (39) we easily find that the Dirichlet series representing  $\mathcal{G}_\alpha(s)$  is absolutely convergent for

$$\operatorname{Re} s > \max(1/2, \log_2(\alpha / \lfloor \alpha \rfloor)).$$

To prove the desired size estimate for  $\Phi_\alpha(n)$ , we require the following simple estimates.

**Lemma 3.5.** *If  $\alpha \geq 1$  and  $0 \leq x < \lfloor \alpha \rfloor / \alpha$ , then*

$$(40) \quad G_{\alpha+1}(x) \leq G_\alpha(x).$$

Moreover,  $G_\alpha$  enjoys uniform estimates for  $0 \leq x \leq 1/2$ ,

$$1 \leq G_\alpha(x) \leq 1 + x^2 \begin{cases} 16(\alpha-1)/(2-\alpha)^3, & 1 \leq \alpha < 2, \\ 384, & \alpha \geq 2. \end{cases}$$

*Proof.* To prove (40), we look at the Taylor expansion of the logarithm

$$\log(G_\alpha(x)) = \sum_{j=2}^{\infty} \frac{x^j}{j} \left( \lfloor \alpha \rfloor \left( \frac{\alpha}{\lfloor \alpha \rfloor} \right)^j - \alpha \right).$$

It is sufficient to show that  $C_j(\alpha+1) \leq C_j(\alpha)$ , where

$$C_j(\alpha) := \lfloor \alpha \rfloor \left( \frac{\alpha}{\lfloor \alpha \rfloor} \right)^j - \alpha.$$

Clearly  $C_j(\lfloor \alpha \rfloor) = C_j(\lfloor \alpha + 1 \rfloor) = 1$ . We set  $\alpha = \lfloor \alpha \rfloor + t$  for  $0 \leq t < 1$ , and differentiate to find that

$$\frac{d}{dt} C_j(\alpha) = j \left( \frac{\alpha}{\lfloor \alpha \rfloor} \right)^{j-1} - 1 \geq j \left( \frac{\alpha+1}{\lfloor \alpha+1 \rfloor} \right)^{j-1} - 1 = \frac{d}{dt} C_j(\alpha+1).$$

The lower bound in the second statement is just Bernoulli's inequality,

$$\left( 1 - \frac{\alpha}{\lfloor \alpha \rfloor} x \right)^{\lfloor \alpha \rfloor / \alpha} \leq 1 - x.$$

The upper bounds can be computed with Taylor's theorem. By (40), we only need to consider  $1 \leq \alpha < 2$  and  $\alpha \geq 2$ . The precise value of the constants are unimportant; we have obtained ours by rather coarse estimates.  $\square$

Using standard techniques (see e.g. [50, Ch. II.5]), we now deduce that the average order of  $\Phi_\alpha(n)$  is the same as the average order of  $d_\alpha(n)$  given by (38).

**Lemma 3.6.** *Let  $\Phi_\alpha(n)$  denote the weight (32) for fixed  $\alpha \geq 1$ . Then*

$$\frac{1}{x} \sum_{n \leq x} \Phi_\alpha(n) = \frac{\mathcal{G}_\alpha(1)}{\Gamma(\alpha)} (\log x)^{\alpha-1} + O((\log x)^{\alpha-2}).$$

**3.4. A theorem on local embedding for  $0 < p < 2$ .** We will now use Theorem 3.4 and Lemma 3.6 to prove an embedding theorem for the Hardy spaces of Dirichlet series  $\mathcal{H}^p$ , when  $p < 2$ . Let  $\mathcal{T}$  denote the following conformal map from  $\mathbb{D}$  to  $\mathbb{C}_{1/2}$ ,

$$\mathcal{T}(z) := \frac{1}{2} + \frac{1-z}{1+z}.$$

For  $\alpha > 1$ , define the conformally invariant Bergman space  $A_{\alpha,i}^2(\mathbb{C}_{1/2})$  as the space of analytic functions  $f$  in  $\mathbb{C}_{1/2}$  such that  $f \circ \mathcal{T}$  is in  $A_\alpha^2(\mathbb{D})$ . In particular, set

$$\|f\|_{A_{\alpha,i}^2(\mathbb{C}_{1/2})} = \|f \circ \mathcal{T}\|_{A_\alpha^2(\mathbb{D})} = \left( \int_{\mathbb{C}_{1/2}} |f(s)|^2 (\alpha-1) \left( \sigma - \frac{1}{2} \right)^{\alpha-2} \frac{4^{\alpha-1} dm(s)}{\pi |s+1/2|^{2\alpha}} \right)^{\frac{1}{2}}.$$

We are able to extend [6, Thm. 1] from  $1 \leq p < 2$  to  $p < 2$  using Theorem 3.4. Note that this is a Dirichlet series version of (23) in the half-plane  $\mathbb{C}_{1/2}$ .

**Corollary 3.7.** *Let  $0 < p < 2$ . There is a constant  $C_p \geq 1$  such that*

$$\|f\|_{A_{2/p,i}^2(\mathbb{C}_{1/2})} \leq C_p \|f\|_{\mathcal{H}^p}$$

*for every  $f \in \mathcal{H}^p$ . The parameter  $\alpha = 2/p$  is optimal.*

*Proof.* Define  $\mathcal{H}_\alpha$  as the Hilbert space of Dirichlet series  $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  that satisfy

$$\|f\|_{\mathcal{H}_\alpha} := \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{\Phi_\alpha(n)} \right)^{\frac{1}{2}} < \infty.$$

Here it is crucial that  $\Phi_\alpha$  is strictly positive. By Lemma 3.6 and [35, Thm. 1] it follows that there is some  $C_\alpha$  such that

$$\|f\|_{A_{\alpha,i}^2(\mathbb{C}_{1/2})} \leq C_\alpha \|f\|_{\mathcal{H}_\alpha},$$

whenever  $\alpha > 1$ . The proof of the first statement is completed using (34). For the proof that  $\alpha = 2/p$  is optimal, we can follow the argument given in the proof of [6, Thm. 1]. We set

$$f_{p,\varepsilon}(s) = \zeta^{2/p}(1/2 + \varepsilon + s) = \sum_{n=1}^{\infty} \frac{d_{2/p}(n)}{n^{1/2+\varepsilon}} n^{-s},$$

which satisfies

$$f_{p,\varepsilon}(s) = \left( \frac{1}{1/2 + \varepsilon + s - 1} \right)^{2/p} + O(|1/2 + \varepsilon + s - 1|^{-(2/p-1)})$$

when  $1/2 < \operatorname{Re} s = \sigma < 1$  and  $0 < \operatorname{Im} s = t < 1$ . Then clearly

$$\|f_{p,\varepsilon}\|_{A_{\alpha,i}^2(\mathbb{C}_{1/2})}^2 \gg \int_{1/2}^1 \int_0^1 \left| \frac{1}{\sigma - 1/2 + \varepsilon + it} \right|^{\frac{4}{p}} \left( \sigma - \frac{1}{2} \right)^{\alpha-2} dt d\sigma \gg \varepsilon^{\alpha-4/p}.$$

Since  $\|f_{p,\varepsilon}\|_{\mathcal{H}^p}^2 \asymp \varepsilon^{-2/p}$ , we get that  $\alpha - 4/p \geq -2/p$  is necessary.  $\square$

**3.5. Fractional primitives of  $\zeta(s)$  and duality.** It was asked in [13, Sec. 5] whether the primitive of the half-shift of the Riemann zeta function

$$\varphi(s) := 1 + \sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \log n} n^{-s}$$

defines a bounded linear functional on  $\mathcal{H}^1$ , or equivalently: Is there a constant  $C$  such that

$$(41) \quad \left| a_1 + \sum_{n=2}^N \frac{a_n}{\sqrt{n} \log n} \right| \leq C \|f\|_{\mathcal{H}^p}$$

for every Dirichlet polynomial  $f(s) = \sum_{n=1}^N a_n n^{-s}$  when  $p = 1$ ? Clearly, (41) is satisfied if  $p = 2$ , and it was shown in [6] that (41) holds whenever  $p > 1$ .

It was also demonstrated in [6] that  $\varphi$  is in  $\mathcal{H}^p$  if and only if  $p < 4$ . We are still not able to answer the original question from [13, Sec. 5], but we will prove some complementary results that shed more light on this and related questions about duality.

For  $\beta > 0$ , consider the following fractional primitives of the half-shift of the Riemann zeta function:

$$(42) \quad \varphi_{\beta}(s) := 1 + \sum_{n=2}^{\infty} \frac{1}{\sqrt{n} (\log n)^{\beta}} n^{-s}.$$

We are interested in the following questions.

- (a) For which  $\beta > 0$  is  $\varphi_{\beta}$  in  $\mathcal{H}^p$ , when  $p \geq 2$ ?
- (b) For which  $\beta > 0$  is  $\varphi_{\beta}$  in  $(\mathcal{H}^p)^*$ , when  $p \leq 2$ ?

Before proceeding, let us clarify question (b). The linear functional generated by  $\varphi_{\beta}$  can be expressed as

$$\langle f, \varphi_{\beta} \rangle_{\mathcal{H}^2} := a_1 + \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{n} (\log n)^{\beta}},$$

when  $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ . We say that the linear functional generated by  $\varphi_{\beta}$  acts boundedly on  $\mathcal{H}^p$ , or equivalently that  $\varphi_{\beta}$  is in  $(\mathcal{H}^p)^*$ , if there is a constant  $C > 0$  such that

$$|\langle f, \varphi_{\beta} \rangle_{\mathcal{H}^2}| \leq C \|f\|_{\mathcal{H}^p}$$

for every Dirichlet polynomial  $f$ . Our result is:

**Theorem 3.8.** *Suppose that  $\beta > 0$ .*

- (a) *Let  $p \geq 2$ . Then  $\varphi_{\beta}$  is in  $\mathcal{H}^p$  if and only if  $\beta > p/4$ .*
- (b) *Let  $p \leq 2$ . If  $\beta > 1/p$  then  $\varphi_{\beta}$  is in  $(\mathcal{H}^p)^*$  and if  $\beta < 1/p$  then  $\varphi_{\beta}$  is not in  $(\mathcal{H}^p)^*$ .*

It is well-known that the dual space  $(\mathcal{H}^p)^*$  for  $1 < p < \infty$  is not equal to  $\mathcal{H}^q$  with  $1/p + 1/q = 1$  (see [44, Sec. 3]). Theorem 3.8 provides additional examples illustrating this fact.

Before proving Theorem 3.8, we note that only the case  $\beta = 1$  in Theorem 3.8 can be proved completely using results from [6, 8, 45], and that (33) or (34) are required for either (a) or (b) when  $\beta \neq 1$ .

*Proof of Theorem 3.8 (a).* To begin with, we notice that (33) implies that

$$\|\varphi_\beta\|_{\mathcal{H}^p}^2 \leq 1 + \sum_{n=2}^{\infty} \frac{\Phi_{p/2}(n)}{n(\log n)^{2\beta}}.$$

The series on the right-hand side is convergent when  $2\beta > p/2$ , by Lemma 3.6 and Abel summation, and we have thus proved that  $\varphi_\beta$  is in  $\mathcal{H}^p$  whenever  $\beta > p/4$ .

To settle the case  $\beta = p/4$ , we set  $k = [p]$ ,  $q = p/k$ , and

$$\log^* n = \begin{cases} \log n, & n > 1 \\ 1, & n = 1, \end{cases}$$

and use (35) to the effect that

$$\begin{aligned} \|\varphi_\beta\|_{\mathcal{H}^p}^p &= \|\varphi_\beta^k\|_{\mathcal{H}^q}^q \geq \left( \sum_{n=1}^{\infty} \frac{|\mu(n)|}{d_{2/q}(n)} \frac{1}{n} \left| \sum_{n_1 \dots n_k = n} \frac{1}{(\log^* n_1)^\beta \dots (\log^* n_k)^\beta} \right|^2 \right)^{\frac{r}{2}} \\ &\geq \left( \sum_{n=2}^{\infty} \frac{|\mu(n)|}{d_{2/q}(n)} \frac{[d_k(n)]^2}{n(\log n)^{2k\beta}} \right)^{\frac{r}{2}} = \left( \sum_{n=2}^{\infty} \frac{|\mu(n)| d_{p[p]/2}(n)}{n(\log n)^{p[p]/2}} \right)^{\frac{r}{2}}, \end{aligned}$$

where we used thrice that  $|\mu(n)| d_\alpha(n) = |\mu(n)| \alpha^{\Omega(n)}$ . To see that the final series is divergent, we use Abel summation and the fact that

$$\frac{1}{x} \sum_{n \leq x} |\mu(n)| d_\alpha(n) = C_\alpha (\log x)^{\alpha-1} + O((\log x)^{\alpha-2}),$$

which follows at once from standard techniques since

$$\sum_{n=1}^{\infty} |\mu(n)| d_\alpha(n) n^{-s} = \prod_{j=1}^{\infty} (1 + \alpha p_j^{-s}). \quad \square$$

*Proof of Theorem 3.8 (b).* The first statement follows from (34), since the Cauchy–Schwarz inequality gives that

$$|\langle f, \varphi_\beta \rangle_{\mathcal{H}^2}| \leq \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{\Phi_{2/p}(n)} \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \frac{\Phi_{2/p}(n)}{n(\log n)^{2\beta}} \right)^{\frac{1}{2}}.$$

Abel summation again gives that the final sum is convergent if  $2\beta > 2/p$ .

For the second part, suppose that  $\beta < 1/p$  and set

$$f(s) = \left( \prod_{p_j \leq N} \frac{1}{1 - p_j^{-1/2-s}} \right)^{2/p}.$$

Clearly,  $\|f\|_{\mathcal{H}^p} \asymp (\log N)^{1/p}$ . We use Abel summation and (38) and find that

$$\langle f, \varphi_\beta \rangle_{\mathcal{H}^2} \geq \sum_{n=2}^N \frac{d_{2/p}(n)}{n(\log n)^\beta} \asymp (\log N)^{2/p-\beta}.$$

We conclude that

$$\frac{\langle f, \varphi_\beta \rangle_{\mathcal{H}^2}}{\|f\|_{\mathcal{H}^p}} \asymp (\log N)^{1/p-\beta}$$

is unbounded as  $N \rightarrow \infty$ , since by assumption  $\beta < 1/p$ .  $\square$

The proof of Theorem 3.8 (b) does not provide any insight into the critical exponent  $\beta = 1/p$ , except for the trivial case  $p = 2$ . The final part of this subsection is devoted to some observations on this interesting problem. We begin by considering the corresponding problem for Hardy spaces on the unit disc. To this end, we introduce

$$(43) \quad \psi_\beta(z) := \sum_{j=0}^{\infty} \frac{z^j}{(j+1)^\beta},$$

which are fractional primitives of  $(1-z)^{-1}$ , a function which plays the same role as  $\zeta(s)$  in the theory of Hardy spaces in the unit disc. Equivalently, one could consider the linear functional with weights given by

$$\text{Beta}\left(\beta, \frac{j+1}{2}\right) \asymp_\beta (j+1)^\beta.$$

These weights are sometimes more convenient, due to the fact that the associated functional  $L_\beta$  admits the integral representation

$$(44) \quad L_\beta(f) := \int_0^1 f(r) 2(1-r^2)^{\beta-1} dr.$$

We compile the following result:

**Theorem 3.9.** *Let  $\psi_\beta$  be as in (43). Then*

- (a) *If  $1 < p < \infty$ , then  $\psi_\beta$  is in  $(H^p(\mathbb{D}))^* = H^{p/(p-1)}(\mathbb{D})$  if and only if  $\beta > 1/p$ .*
- (b) *If  $p \leq 1$ , then  $\psi_\beta$  is in  $(H^p(\mathbb{D}))^*$  if and only if  $\beta \geq 1/p$ . Moreover, if  $\beta \geq 1$ , then  $\psi_\beta$  is in  $H^p(\mathbb{D})$  for every  $p < \infty$ .*

*Proof.* We begin with (a). That  $(H^p(\mathbb{D}))^* = H^{p/(p-1)}(\mathbb{D})$  for  $1 < p < \infty$  is well-known (see [19]). We will investigate when  $\psi_\beta$  is in  $H^{p/(p-1)}(\mathbb{D})$ . To do this, we use a result of Hardy and Littlewood [23]: If  $f(z) = \sum_{j=0}^{\infty} a_j z^j$  has positive and decreasing coefficients and  $1 < q < \infty$ , then

$$\|f\|_{H^q(\mathbb{D})} \asymp_q \left( \sum_{j=0}^{\infty} (j+1)^{q-2} a_j^q \right)^{\frac{1}{q}}.$$

Setting  $q = p/(p-1)$  we find that

$$\|\psi_\beta\|_{H^q(\mathbb{D})}^q \asymp_q \sum_{j=0}^{\infty} (j+1)^{\frac{p}{p-1}(1-\beta)-2},$$

which is finite if and only if  $\beta > 1/p$ .

For (b), we begin with the case  $\beta = 1$ . A stronger version of our statement can be found in [19, Thm. 4.5]. It is also clear that since  $\psi_1$  is in  $(H^1(\mathbb{D}))^*$ ,  $\psi_1$  is in  $H^q(\mathbb{D})$  for every  $p < \infty$ .

To investigate the case  $p < 1$ , we require the main result in [20] for which we refer to [19]. We conclude that  $\psi_\beta \in (H^p(\mathbb{D}))^*$  if and only if  $\beta \leq 1/p$  by combining [19, Thm. 7.5] with [19, Ex. 1 and Ex. 3 on p. 90]. If  $\beta < 1$ , then  $\psi_\beta$  is a bounded function, so  $\psi_\beta$  is in  $H^p(\mathbb{D})$  for every  $p < \infty$ .  $\square$

In analogy with Theorem 3.9, we offer the following conjecture.



*Conjecture.* Let  $p \leq 2$ . The Dirichlet series  $\varphi_{1/p}$  from (42) defines a bounded linear functional on  $\mathcal{H}^p$  if and only if  $p \leq 1$ .

Let us now explain how this conjecture is related to another open problem for Hardy spaces of Dirichlet series. Define the space  $H_1^p(\mathbb{C}_{1/2})$  in the same way as  $A_{\alpha,i}^2(\mathbb{C}_{1/2})$  from Corollary 3.7. A computation gives that

$$\|f\|_{H_1^p(\mathbb{C}_{1/2})}^p = \frac{1}{\pi} \int_{-\infty}^{\infty} |f(1/2 + it)|^p \frac{dt}{1+t^2}.$$

An equivalent formulation of the embedding problem mentioned in Section 2.5 is the following. Is there is a constant  $C_p > 0$  such that

$$(45) \quad \|f\|_{H_1^p(\mathbb{C}_{1/2})}^p \leq C_p \|f\|_{\mathcal{H}^p}^p$$

holds for every Dirichlet polynomial  $f$ ? As mentioned in Section 2.5, the embedding (45) is known to hold only when  $p$  is an even integer. See [10] for a simple proof of this fact, which also gives the optimal constant<sup>2</sup>  $C_p = 2$ . Note that (45) is a stronger statement than Corollary 3.7, since from (23) we get that

$$\|f\|_{A_{2/p,i}^2(\mathbb{C}_{1/2})} \leq C_p \|f\|_{H_1^p(\mathbb{C}_{1/2})}$$

for  $0 < p < 2$ . The linear functional generated by  $\varphi_\beta$  can also be expressed as

$$(46) \quad \langle f, \varphi_\beta \rangle_{\mathcal{H}^2} = a_1 + \int_{1/2}^{\infty} (f(\sigma) - a_1) \left( \sigma - \frac{1}{2} \right)^{\beta-1} \frac{d\sigma}{\Gamma(\beta)}.$$

Translating (44) from  $\mathbb{D}$  to  $\mathbb{C}_{1/2}$  we find that

$$\tilde{L}_{1/p}(F) := \int_{1/2}^1 F(\sigma) \left( \sigma - \frac{1}{2} \right)^{1/p-1} d\sigma$$

defines a bounded linear functional on  $H_1^p(\mathbb{C}_{1/2})$  if and only if  $p \leq 1$ . Note that the contribution in (46) for  $\sigma \geq 1$  can be handled by trivial estimates. Hence we conclude that if (45) holds for  $p \leq 1$ , then (46) (and hence  $\varphi_{1/p}$ ) defines a bounded linear functional on  $\mathcal{H}^p$ .

#### 4. COEFFICIENT ESTIMATES: WEIGHTED $\ell^\infty$ BOUNDS

We now turn to weighted  $\ell^\infty$  estimates of the coefficient sequence  $(a_n)_{n \geq 1}$  for elements  $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  in  $\mathcal{H}^p$ . Phrased differently, we are interested in estimating the norm of the linear functional  $f \mapsto a_n$  for every  $n \geq 1$ , i.e., the quantity

$$\mathcal{C}(n, p) := \sup_{\|f\|_p=1} |a_n|.$$

When  $p \geq 1$ ,  $a_n$  can be expressed as a Fourier coefficient, implying that this norm is trivially 1 for all  $n$ . We will therefore mainly be concerned with the case  $0 < p < 1$ .

The first observation we make is that, again, it suffices to deal with the one-dimensional situation because the general estimates will appear by multiplicative extension. Before we prove this claim, we recall what is known about the coefficients of  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  in  $H^p(\mathbb{D})$  when  $0 < p < 1$ . For  $0 < p < \infty$  and  $k \geq 1$ , we set

$$(47) \quad C(k, p) := \sup \left\{ \frac{|f^{(k)}(0)|}{k!} : \|f\|_{H^p(\mathbb{D})} = 1 \right\}.$$

<sup>2</sup>The proof given in [10] that  $C_2 = 2$  extends effortlessly to show that  $C_p = 2$  when  $p$  is an even integer.

By a classical result [19, p. 98],  $C(k, p) \ll k^{1/p-1}$  and  $a_k = o(k^{1/p-1})$  for an individual function in  $H^p(\mathbb{D})$  when  $0 < p < 1$ . By a normal family argument, there are extremal functions  $f_k$  in  $H^p(\mathbb{D})$  for (47).

Turning to the multiplicative extension, we begin by noting that it suffices to consider an arbitrary polynomial

$$F(z) = \sum_{\kappa} c_{\kappa} z^{\kappa}$$

on  $\mathbb{T}^{\infty}$  and to estimate the size of  $c_{\kappa}$  for an arbitrary multi-index  $\kappa = (\kappa_1, \dots, \kappa_m, 0, 0, \dots)$ . Recall that  $A_m F$  denotes the  $m$ th Abschnitt of  $F$ . For  $0 < p < 1$  we use (5) to find that

$$\begin{aligned} |c_{\kappa}|^p &= \left| \int_{\mathbb{T}^m} A_m F(z) \overline{z_1}^{\kappa_1} \dots \overline{z_m}^{\kappa_m} d\mu_m \right|^p \\ &\leq C(\kappa_m, p)^p \int_{\mathbb{T}} \left| \int_{\mathbb{T}^{m-1}} A_m F(z) \overline{z_1}^{\kappa_1} \dots \overline{z_{m-1}}^{\kappa_{m-1}} d\mu_{m-1} \right|^p d\mu_1 \\ &\leq C(\kappa_1, p)^p \dots C(\kappa_m, p)^p \|A_m F\|_p^p \\ &\leq C(\kappa_1, p)^p \dots C(\kappa_m, p)^p \|F\|_p^p. \end{aligned}$$

This is a best possible estimate because if  $f_k$  in  $H^p(\mathbb{D})$  satisfies  $|a_k|/\|f_k\|_p = C(k, p)$ , then clearly the function

$$\prod_{j=1}^m f_{\kappa_j}(z_j)$$

will be extremal with respect to the multi-index  $\kappa = (\kappa_1, \dots, \kappa_m)$ . Hence we conclude that  $n \mapsto \mathcal{C}(n, p)$  is a multiplicative function that takes the value  $C(k, p)$  at  $n = p_j^k$  for every prime  $p_j$ .

To the best of our knowledge, the exact values of  $C(k, p)$  from (47) have not been computed previously for any  $k \geq 1$  when  $0 < p < 1$ , and we have therefore made an effort to improve this situation. We begin with the case  $k = 1$  which is settled by the following theorem.

**Theorem 4.1.** *We have*

$$(48) \quad C(1, p) = 1 \quad \text{if } p \geq 1, \quad \text{and} \quad C(1, p) = \sqrt{\frac{2}{p}} \left(1 - \frac{p}{2}\right)^{\frac{1}{p} - \frac{1}{2}} \quad \text{if } 0 < p < 1.$$

*The corresponding extremals (modulo the trivial modifications  $f(z) \mapsto e^{i\theta_1} f(e^{i\theta_2} z)$ ) are*

- (a)  $f(z) = z$  for  $p > 1$ ;
- (b) the family  $f_a(z) = (a + \sqrt{1 - a^2} z)(\sqrt{1 - a^2} + az)$  with  $0 \leq a \leq 1$  for  $p = 1$ ;
- (c)  $f(z) = (\sqrt{1 - p/2} + z\sqrt{p/2})^{2/p}$  for  $0 < p < 1$ .

*Proof.* As already pointed out, it is obvious that  $C(1, p) = 1$  when  $p \geq 1$ . The uniqueness of the extremal function for  $p > 1$  is immediate by the strict convexity of the unit ball of  $L^p(\mathbb{T})$ .

To find the extremal functions when  $p = 1$ , we start from the fact that functions  $f$  in the unit ball of  $H^1(\mathbb{T})$  can equivalently be written in the form  $f = gh$ , where  $h, g$  are in the unit ball of  $H^2(\mathbb{T})$ . Writing  $g(z) = \sum_{k=0}^{\infty} g_k z^k$  and  $h(z) = \sum_{k=0}^{\infty} h_k z^k$ , our task is to maximize

$$f'(0) = g_0 h_1 + g_1 h_0,$$

under the sole condition that  $\sum_{k=0}^{\infty} |g_k|^2 = 1$  and  $\sum_{k=0}^{\infty} |h_k|^2 = 1$ . By the Cauchy–Schwarz inequality, we must have  $|g_0|^2 + |g_1|^2 = |h_0|^2 + |h_1|^2 = 1$  and also  $(g_0, g_1) = \lambda(h_1, h_0)$  for a unimodular constant  $\lambda$ . We may choose  $(g_0, g_1)$  as an arbitrary unit vector, and hence we get the stated extremals.

We turn to the case  $0 < p < 1$ . By invoking the inner-outer factorization of  $f$ , we may write an arbitrary element  $f$  with in the unit ball of  $H^p(\mathbb{D})$  equivalently as  $f = gh^{2/p-1}$ , where  $g, h$  are in the unit ball of  $H^2(\mathbb{T})$  and  $h$  has no zeros in  $\mathbb{D}$ . We denote the coefficients of  $g$  and  $h$  as before. By applying a suitable transformation  $f(z) \mapsto e^{i\theta_1} f(e^{i\theta_2} z)$ , we may assume that  $h_0, h_1 \geq 0$ , and, moreover, that  $f(0) = g_0 h_0^{2/p-1}$ , where  $h_0^{2/p-1} \geq 0$  is chosen to be real and nonnegative. Hence

$$C(1, p) = \sup \left( h_0^{2/p-1} g_1 + \left( \frac{2}{p} - 1 \right) h_0^{2/p-2} h_1 g_0 \right),$$

where the supremum is over all pairs  $(g_0, g_1)$  with  $|g_0|^2 + |g_1|^2 = 1$  and pairs of nonnegative numbers  $(h_0, h_1)$  with  $h_0^2 + h_1^2 = 1$  and  $h_0 \geq h_1$  since  $h$  is zero-free.

The maximum occurs when  $(g_0, g_1)$  is a multiple of  $((\frac{2}{p} - 1)h_0^{2/p-2}h_1, h_0^{2/p-1})$  and hence

$$C(1, p)^2 = \max_{h_0^2 + h_1^2 = 1, h_0 \geq h_1 \geq 0} \left( h_0^{4/p-2} + \left( \frac{2}{p} - 1 \right)^2 h_0^{4/p-4} h_1^2 \right).$$

Suppressing the condition  $h_0 \geq h_1$ , we find that

$$(49) \quad C(1, p)^2 \leq \max_{x \in [0, 1]} \left( x^{4/p-2} + \left( \frac{2}{p} - 1 \right)^2 x^{4/p-4} (1 - x^2) \right) = \frac{2}{p} \left( 1 - \frac{p}{2} \right)^{2/p-1}$$

by an elementary calculus argument. Since the solution to the extremal problem in (49) corresponds to  $h_0 = \sqrt{1 - p/2}$ , we also have  $h_0 \geq h_1$ , and the inequality sign in (49) can therefore in fact be replaced by an equality sign.  $\square$

For future reference, we notice that the following asymptotic estimates hold:

$$(50) \quad C(1, p) = \begin{cases} 1 + (1 - \log 2)(1 - p) + O((1 - p)^2), & p \nearrow 1 \\ \frac{1}{\sqrt{p}} \cdot (\sqrt{2/e} + O(p)), & p \searrow 0. \end{cases}$$

For  $k \geq 2$ , the method used in the preceding proof will lead to a similar finite-dimensional extremal problem. The solution to this problem is plain for all  $k \geq 2$  when  $p = 1$ , but in the range  $0 < p < 1$ , the complexity increases notably with  $k$ , and we have made no attempt to deal with it. Instead, we supply (non-optimal) estimates obtained from the Cauchy integral formula and Lemma 3.2 (Weissler's inequality).

**Lemma 4.2.** *Suppose that  $0 < p < 1$  and  $k \geq 1$ . Then*

$$C(k, p) \leq \min_{p \leq x < 1} x^{-k/2} (1 - x)^{1/x-1/p}.$$

*Proof.* Suppose that  $f(z) = \sum_{k=0}^{\infty} c_k z^k$  is in  $H^p(\mathbb{D})$  with  $\|f\|_p = 1$ . Then, by Cauchy's formula,

$$|c_k| \leq \frac{1}{2\pi r} \int_{|z|=r} |z^{-k} f(z)| |dz|$$

for every  $r$ ,  $0 < r < 1$ . Using the pointwise estimate  $|f(z)| \leq (1 - |z|^2)^{-1/p} \|f\|_p$ , we therefore find that

$$|c_k| \leq r^{-k} ((1 - r^2)^{1/p})^{q-1} \|f\|_p^{1-q} \int_{\mathbb{T}} |f(rz)|^q d\mu(z)$$

whenever  $0 < r < 1$  and  $0 < q < 1$ . Choosing  $p < q \leq 1$  and  $r^2 = p/q$  and invoking Lemma 3.2, we obtain the desired result.  $\square$

We notice that (26) of Theorem 3.3 yields the alternate bound<sup>3</sup>

$$(51) \quad C(k, p) \leq \sqrt{c_{\lceil 2/p \rceil}(k)}$$

which is useful when  $p$  is close to 0.

We will now use the information gathered about to prove a result about the maximal order of the multiplicative function  $n \mapsto \mathcal{C}(n, p)$ . To begin with, we notice that, by Theorem 4.1,

$$\mathcal{C}(n, p) = C(1, p)^{\omega(n)}$$

when  $n$  is a square-free number and hence

$$(52) \quad \limsup_{\mu(n) \neq 0, n \rightarrow \infty} \frac{\log \mathcal{C}(n, p)}{\log n / \log \log n} = \log C(1, p)$$

since

$$\limsup_{\mu(n) \neq 0, n \rightarrow \infty} \frac{\log \omega(n)}{\log n / \log \log n} = 1.$$

It seems reasonable to expect that the lim sup in (52) is unchanged if we drop the restriction that  $\mu(n) \neq 0$ . The next theorem is as close as we have been able to get to confirming this conjecture, based on our general bounds for  $C(k, p)$ .

**Theorem 4.3.** *Assume that  $0 < p < 1$ . Then*

$$0 < \limsup_{n \rightarrow \infty} \frac{\log \mathcal{C}(n, p)}{\log n / \log \log n} < \infty.$$

Moreover,

$$\limsup_{n \rightarrow \infty} \frac{\log \mathcal{C}(n, p)}{\log n / \log \log n} = \begin{cases} \frac{1}{2} |\log p| (1 + O(p)), & p \searrow 0 \\ c_p (1 - p), & p \nearrow 1, \end{cases}$$

where  $1 - \log 2 + O(1 - p) \leq c_p \leq 1/2 + O(1 - p)$ .

*Proof.* The general lower bound for the lim sup follows from (52), while the lower bounds

$$\limsup_{n \rightarrow \infty} \frac{\log \mathcal{C}(n, p)}{\log n / \log \log n} \geq \begin{cases} \frac{1}{2} |\log p| (1 + O(p)), & p \searrow 0 \\ (1 - \log 2)(1 - p) + O((1 - p)^2), & p \nearrow 1 \end{cases}$$

follow from (52) along with (50). To get an upper bound for the lim sup when  $p \searrow 0$ , we use (51) and the fact that

$$\limsup_{n \rightarrow \infty} \frac{\log d_\alpha(n)}{\log n / \log \log n} = \log \alpha.$$

Trivially, (51) also gives a general upper bound for the lim sup.

To get an upper bound for the lim sup when  $p \nearrow 1$ , we argue as follows. Set

$$n = \prod_j p_j^{\kappa_j}.$$

For  $\kappa_j \leq 1/(1 - p)$ , we set  $x = p$  in Lemma 4.2 and get

$$(53) \quad C(\kappa_j, p) \leq p^{-\kappa_j/2}.$$

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<sup>3</sup>Notice that the bound  $C(k, p) \leq \sqrt{\varphi_{2/p}(k)}$  is of no interest in this context because  $\varphi_{2/p}(k)$  grows exponentially with  $k$  when  $2/p$  is not an integer.

We note that

$$(54) \quad \begin{aligned} \sum_{j:\kappa_j \leq 1/(1-p)} \kappa_j &\leq \frac{1}{1-p} \sum_{j \leq \log n / (\log \log n)^2} 1 + \frac{(1+o(1))}{\log \log n} \sum_{j:\kappa_j \leq 1/(1-p)} \kappa_j \log p_j \\ &= \frac{\log n}{(1-p)(\log \log n)^2} + \frac{(1+o(1))}{\log \log n} \sum_{j:\kappa_j \leq 1/(1-p)} \kappa_j \log p_j. \end{aligned}$$

For  $\kappa_j > 1/(1-p)$ , we set  $x = 1 - (1-p)/\kappa_j$  in Lemma 4.2 so that

$$(55) \quad C(\kappa_j, p) \leq \left(1 - \frac{(1-p)}{\kappa_j}\right)^{-\kappa_j/2} \cdot \left(\frac{1-p}{\kappa_j}\right)^{1-1/p} \leq e^{1-p} \kappa_j^{2(1/p-1)}.$$

We observe that, given  $\varepsilon > 0$ , we will have if  $p$  is close enough to 1, then

$$(56) \quad \begin{aligned} \sum_{j:\kappa_j \geq 1/(1-p)} \log \kappa_j &\leq \log(\log n / \log 2) \sum_{j \leq \log n / (\log \log n)^3} 1 + \frac{\varepsilon}{\log \log n} \sum_{j:\kappa_j > 1/(1-p)} \kappa_j \log p_j \\ &\leq \frac{(1+o(1)) \log n}{(\log \log n)^2} + \frac{\varepsilon}{\log \log n} \sum_{j:\kappa_j > 1/(1-p)} \kappa_j \log p_j \end{aligned}$$

if  $p$  is close enough to 1. Putting (54) into (53) and (56) into (55), respectively, we find that

$$\begin{aligned} \log \mathcal{C}(n, p) &= \sum_j \log C(\kappa_j, p) = \sum_{j:\kappa_j \leq 1/(1-p)} C(\kappa_j, p) + \sum_{j:\kappa_j > 1/(1-p)} C(\kappa_j, p) \\ &\leq o(1) \frac{\log n}{\log \log n} + \frac{|\log p|}{2 \log \log n} \sum_{j:\kappa_j \leq 1/(1-p)} \kappa_j \log p_j + \frac{2(1/p-1)\varepsilon}{\log \log n} \sum_{j:\kappa_j > 1/(1-p)} \kappa_j \log p_j \end{aligned}$$

holds for arbitrary  $\varepsilon > 0$ , if  $p$  is close enough to 1. Choosing  $\varepsilon < 1/4$ , we obtain the desired upper bound for the lim sup.  $\square$

We mention finally a consequence of the classical bound  $C(k, p) \ll k^{1/p-1}$  that was already used in [9], related to the decomposition of a holomorphic function on  $\mathbb{T}^\infty$  into a sum of holomorphic functions with homogeneous power series. Thus we are interested in the orthogonal projection of a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

onto the space of  $m$ -homogeneous functions, namely

$$P_m f(s) := \sum_{\Omega(n)=m} a_n n^{-s}.$$

The result in question, to be used in Subsection 6.3, is as follows.

**Lemma 4.4.** *Suppose that  $0 < p < \infty$ . Then*

$$\|P_m f\|_{\mathcal{H}^p} \leq \begin{cases} \|f\|_{\mathcal{H}^p}, & p \geq 1, \\ \sqrt{e}(m+1)^{1/p-1} \|f\|_{\mathcal{H}^p}, & 0 < p < 1 \end{cases}$$

*holds for every  $f$  in  $\mathcal{H}^p$ .*

*Proof.* We may assume that  $f$  is a Dirichlet polynomial, so that  $\mathcal{B}f(z)$  is continuous on  $\mathbb{T}^\infty$ . We introduce the transformation  $wz := (wz_j)$  on  $\mathbb{T}^\infty$ , where  $w$  is a point on the unit circle  $\mathbb{T}$ . We may then write

$$(\mathcal{B}f)(wz) = \sum_{m=0}^{\infty} (\mathcal{B}P_m f)(z) w^m.$$

It follows that we may consider the functions  $(\mathcal{B}P_m f)(z)$  as the coefficients of a function in one complex variable. We set  $k = m$  and  $x = \max(p, 1 - 1/(m+1))$  in Lemma 4.2 and get

$$|(\mathcal{B}fP_m)(z)|^p \leq \begin{cases} \int_{\mathbb{T}} |(\mathcal{B}P_m f)(wz)|^p d\mu_1(w), & p \geq 1, \\ (1 - 1/(m+1))^{-pm/2} (m+1)^{1-p} \int_{\mathbb{T}} |(\mathcal{B}P_m f)(wz)|^p d\mu_1(w), & 0 < p < 1. \end{cases}$$

Integrating this inequality over  $\mathbb{T}^\infty$  with respect to  $\mu_\infty$  and using Fubini's theorem, we obtain the desired estimate.  $\square$

## 5. ESTIMATES FOR THE PARTIAL SUM OPERATOR

Assume that  $f(s) = \sum_{n=1}^\infty a_n n^{-s}$  is a Dirichlet series in  $\mathcal{H}^p$  for some  $p > 0$ . For given  $N \geq 1$ , the partial sum operator  $S_N$  is defined as the map

$$S_N \left( \sum_{n=1}^\infty a_n n^{-s} \right) := \sum_{n=1}^N a_n n^{-s}.$$

It is of obvious interest to try to determine the norm of  $S_N$  when it acts on the Hardy spaces  $\mathcal{H}^p$ . Helson's version of the M. Riesz theorem [30] shows that  $S_N$  is bounded for  $1 < p < \infty$ , and, moreover, its norm is bounded by the norm of the one-dimensional Riesz projection acting on functions in  $H^p(\mathbb{D})$ . Furthermore, by the same argument of Helson [30], we have the following.

**Lemma 5.1.** *Suppose that  $0 < p < 1$ . We have the estimate*

$$\|S_N f\|_{\mathcal{H}^p} \leq \frac{A}{(1-p)} \|f\|_{\mathcal{H}^1}$$

for  $f$  in  $\mathcal{H}^p$ , where  $A$  is an absolute constant.

We refer to [2, Sec. 3], where it is explained how the lemma follows from Helson's general result concerning compact Abelian groups whose dual is an ordered group [30]. See also Sections 8.7.2 and 8.7.6 of [42]. In our case, the dual group in question is the multiplicative group of positive rational numbers  $\mathbb{Q}_+$  which is ordered by the numerical size of its elements. This means that the bound for  $\|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p}$  in the range  $1 < p < \infty$  relies on the additive structure of the positive integers.

When  $0 < p \leq 1$  or  $p = \infty$ , a natural question is to determine the asymptotic growth of the norm  $\|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p}$  when  $N \rightarrow \infty$ . It is known from [4] and [7] that the growth of both  $\|S_N\|_{\mathcal{H}^1 \rightarrow \mathcal{H}^1}$  and  $\|S_N\|_{\mathcal{H}^\infty \rightarrow \mathcal{H}^\infty}$  is of an order lying between  $\log \log N$  and  $\log N$ . We will confine our discussion to the range  $0 < p \leq 1$  and begin with a new result for the case  $p = 1$ .

**Theorem 5.2.** *We have*

$$\log \log N \ll \|S_N\|_{\mathcal{H}^1 \rightarrow \mathcal{H}^1} \ll \frac{\log N}{\log \log N}.$$

*Proof.* Using Hölder's inequality with  $p = (1 + \varepsilon)/\varepsilon$  and  $p' = 1 + \varepsilon$ , we get

$$\|g\|_{\mathcal{H}^1}^{1-\varepsilon} \leq \left( \frac{\|g\|_{\mathcal{H}^2}}{\|g\|_{\mathcal{H}^1}} \right)^{2\varepsilon} \|g\|_{\mathcal{H}^{1-\varepsilon}}^{1-\varepsilon}.$$

Setting  $g = S_N f$  and applying Lemma 5.1, we get

$$\|S_N f\|_1 \leq A \frac{1}{\varepsilon} \left( \frac{\|S_N f\|_2}{\|S_N f\|_1} \right)^{2\varepsilon/(1-\varepsilon)} \|f\|_1.$$

Now we need to understand how large the ratio  $\|f\|_2/\|f\|_1$  can be when  $f$  is a Dirichlet polynomial of length  $N$ . A precise solution to this problem can be found in the recent paper [18]. For



our purpose, the following one-line argument suffices. By Helson's inequality (which is (34) for  $p = 1$ ) and a well-known estimate for the divisor function, we have

$$\|f\|_2 \leq \max_{n \leq N} \sqrt{d(n)} \|f\|_1 \leq e^{c \frac{\log N}{\log \log N}} \|f\|_1$$

for an absolute constant  $c$ . This means that we can choose  $\varepsilon = (\log \log N) / \log N$  so that we get

$$\|S_N\|_1 \ll (\log N) / \log \log N,$$

as desired.

The lower bound is obvious from the classical one-dimensional result: The Bohr lift maps Dirichlet series in  $\mathcal{H}^p$  of the form  $\sum_{k=0}^{\infty} c_k 2^{-ks}$  to functions in  $H^p(\mathbb{D})$ .  $\square$

It is interesting to notice that our improved upper bound relies on both an additive argument (Lemma 5.1) and a multiplicative argument (Theorem 3.4). We now turn to the case  $0 < p < 1$  which will again require a mixture of additive and multiplicative arguments.

**Theorem 5.3.** *Suppose that  $0 < p < 1$ . There are positive constants  $\alpha_p \leq \beta_p$  such that*

$$e^{\alpha_p \frac{\log N}{\log \log N}} \ll \|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p} \ll e^{\beta_p \frac{\log N}{\log \log N}}.$$

Moreover, we have

$$\liminf_{N \rightarrow \infty} \frac{\log \|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p}}{\log N / \log \log N} \geq \begin{cases} \frac{1}{4} |\log p| + O(1), & p \searrow 0 \\ \frac{1}{2} (1 - \log 2)(1 - p) + O((1 - p)^2), & p \nearrow 1 \end{cases}$$

and

$$\limsup_{N \rightarrow \infty} \frac{\log \|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p}}{\log N / \log \log N} \leq \begin{cases} \frac{1}{2} |\log p| + O(1), & p \searrow 0 \\ c(1 - p) + O((1 - p)^2), & p \nearrow 1, \end{cases}$$

where  $c$  is an absolute constant.

We have made no effort to minimize the constant  $c$ , but mention that our proof gives the value  $\log 2$  times the norm of the operator  $f \mapsto f^*$  from  $H^1(\mathbb{D})$  to  $L^1(\mathbb{T})$ , where  $f^*$  is the radial maximal function of  $f$ . Comparing with Theorem 4.3, we notice that  $\log \|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p}$  has essentially the same maximal order as that of  $\log \mathcal{C}(N, p)$ .

We will split the proof of Theorem 5.3 into three parts. We begin with the easiest case.

*Proof of the upper bound in Theorem 5.3, including the asymptotics when  $p \searrow 0$ .* We begin by setting  $\alpha := \lceil 2/p \rceil$  and apply the Hardy–Littlewood inequality from Theorem 3.4:

$$\|S_N f\|_{\mathcal{H}^p} \leq \|S_N f\|_{\mathcal{H}^2} \leq \left( \max_{n \leq N} \sqrt{d_\alpha(n)} \right) \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{d_\alpha(n)} \right)^{\frac{1}{2}} \ll \alpha^{\frac{\log N}{2 \log \log N} (1+o(1))} \|f\|_{\mathcal{H}^{2/\alpha}},$$

where we in the last step used that

$$d_\alpha(n) \leq \alpha^{\frac{\log n}{\log \log n} (1+o(1))}$$

when  $n \rightarrow \infty$ . We conclude by using that  $\|f\|_{\mathcal{H}^{2/\alpha}} \leq \|f\|_{\mathcal{H}^p}$ , which holds because  $2/\alpha \leq p$ . This argument gives both  $\beta_p = \log(1 + 2/p)$ , say, and the desired asymptotic estimate when  $p \searrow 0$ .  $\square$

We need a more elaborate argument to get the right asymptotic behavior when  $p \nearrow 1$ . We prepare for the proof by first establishing an auxiliary result concerning polynomials on  $\mathbb{T}$ . Here we use again the notation  $f_r(z) := f(rz)$ , where  $f$  is an analytic function on  $\mathbb{D}$  and  $r > 0$ .

**Lemma 5.4.** *Suppose that  $0 < p \leq 1$ . There exists an absolute constant  $C$ , independent of  $p$ , such that if  $1 - r = C^{-1/p} n^{-1}$ , then*

$$(57) \quad \|Q\|_{H^p(\mathbb{D})}^p \leq 2 \|Q_r\|_{H^p(\mathbb{D})}^p$$

for every polynomial  $Q(z) = \sum_{k=0}^n c_k z^k$ .

*Proof.* For this proof, we write  $\|Q\|_p = \|Q\|_{H^p(\mathbb{D})}$ . By the triangle inequality for the  $L^p$  quasi-metric, we have

$$(58) \quad \|Q\|_p^p \leq \|Q - Q_r\|_p^p + \|Q_r\|_p^p.$$

Since

$$|Q(z) - Q_r(z)| = \left| \int_{rz}^z Q'(w) dw \right| \leq (1 - r) \max_{0 \leq \rho \leq 1} |Q'(\rho z)|,$$

we find that

$$\|Q - Q_r\|_p^p \leq A(1 - r)^p \|Q'\|_p^p$$

for an absolute constant  $A$  by the  $H^p$  boundedness of the radial maximal function. Using Bernstein's inequality for  $0 < p \leq 1$  [3, 52], we therefore get that

$$\|Q - Q_r\|_p^p \leq A(1 - r)^p n^p \|Q\|_p^p.$$

Returning to (58), we see that we get the desired result by setting  $C = 2A$ .  $\square$

*Proof of the upper bound in Theorem 5.3 when  $p \nearrow 1$ .* Set

$$m = m(N) := \lfloor \log N / (\log \log N)^3 \rfloor$$

and write  $z := (u, v)$  for a point on  $\mathbb{T}^\infty$ , where  $u = (z_1, \dots, z_m)$  and  $v = (z_{m+1}, z_{m+2}, \dots)$ , so that  $u$  corresponds to the first  $m$  primes. Let  $\xi$  and  $\eta$  be complex numbers and set  $\xi u := (\xi z_1, \dots, \xi z_m)$  and  $\eta v = (\eta z_{m+1}, \eta z_{m+2}, \dots)$ . Also, if  $F$  is a function on  $\mathbb{T}^\infty$  and  $0 < r, \rho \leq 1$ , we set  $F_{r,\rho}(z) := F(ru, \rho v)$ .

We will now apply Lemma 5.4 in two different ways. We begin by applying it to the function  $\xi \mapsto (\mathcal{B}S_N f)(\xi u, v)$ , which is a polynomial of degree at most  $\log N / \log 2$ . This gives

$$\int_{\mathbb{T}} |(\mathcal{B}S_N f)(\xi u, v)|^p d\mu(\xi) \leq 2 \int_{\mathbb{T}} |(\mathcal{B}S_N f)(r\xi u, v)|^p d\mu(\xi)$$

for every point  $(u, v)$  and hence

$$\|(\mathcal{B}S_N f)\|_p^p \leq 2 \|(\mathcal{B}S_N f)_{r,1}\|_p^p$$

by Fubini's theorem, with  $1 - r = C^{-1/p} (\log N / \log 2)^{-1}$ . Next, we apply (57) to the function  $\eta \mapsto (\mathcal{B}S_N f)_{r,1}(u, \eta v)$ , which is a polynomial of degree at most  $(1 + o(1)) \log N / \log \log N$ . Hence we find that

$$\|(\mathcal{B}S_N f)\|_p^p \leq 2^2 \|(\mathcal{B}S_N f)_{r,\rho}\|_p^p$$

with  $1 - \rho = C^{-1/p} (1 + o(1)) \log \log N / \log N$ . Applying (57)  $k$  times in this way, we therefore get that

$$(59) \quad \|(\mathcal{B}S_N f)\|_p^p \leq 2^{k+1} \|(\mathcal{B}S_N f)_{r,\rho^k}\|_p^p.$$

We choose  $k$  such that  $\rho^k \leq \sqrt{p}$ , which is done because our plan is to use Lemma 3.2 (Weissler's inequality). Since  $1 - \rho = C^{-1/p} (1 + o(1)) \log \log N / \log N$ , we therefore obtain the requirement that

$$(60) \quad k = \frac{\log p}{2 \log \rho} = |\log p| \cdot (1/2 + o(1)) C^{1/p} \log N / \log \log N.$$

We now apply Lemma 5.1 to the right-hand side of (59), which yields

$$\|(\mathcal{B}S_N f)\|_p \leq K(k, p) \|\mathcal{B}f_{r, \rho^k}\|_1,$$

where

$$(61) \quad K(k, p) := A^p 2^{(k+1)/p} (1-p)^{-1} = A^p (1-p)^{-1} \exp\left(\left(\frac{\log 2}{2} + o(1)\right) |\log p| p^{-1} C^{1/p} \frac{\log N}{\log \log N}\right);$$

here we took into account (60) to get to the final bound for  $K(k, p)$ . Note that, in view of (5), we may assume that  $v$  is a vector of length  $d := \pi(N) - m$ . It follows that

$$\begin{aligned} \|\mathcal{B}S_N f\|_p &\leq K(k, p) \int_{\mathbb{T}^d} \int_{\mathbb{T}^m} |(\mathcal{B}f)(ru, \rho^k v)| d\mu_m(u) d\mu_d(v) \\ &\leq K(k, p) (1-r^2)^{-m(1-p)/p} \int_{\mathbb{T}^d} \left( \int_{\mathbb{T}^m} |(\mathcal{B}f)(u, \rho^k v)|^p d\mu_m(u) \right)^{1/p} d\mu_d(v), \end{aligned}$$

where we in the last step used the Cole–Gamelin estimate (6). Using Minkowski's inequality (36) as before, we thus get

$$\|\mathcal{B}S_N f\|_p^p \leq K(k, p)^p (1-r^2)^{-m(1-p)} \int_{\mathbb{T}^m} \left( \int_{\mathbb{T}^d} |(\mathcal{B}f)(u, \rho^k v)|^p d\mu_d(v) \right)^p d\mu_m(u).$$

We now iterate Weissler's inequality along with Minkowski's inequality  $d$  times in the same way as in the proof of Theorem 3.4 and get the bound

$$\|\mathcal{B}S_N f\|_p \leq K(k, p) (1-r^2)^{-m(1-p)/p} \|f\|_p.$$

Now taking into account our choice of  $r$  and  $m$ , we find that

$$\limsup_{N \rightarrow \infty} \frac{\log \|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p}}{\log N / \log \log N} \leq \limsup_{N \rightarrow \infty} \frac{\log K(k, p)}{\log N / \log \log N}.$$

Using finally (61), we conclude that

$$\limsup_{N \rightarrow \infty} \frac{\log \|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p}}{\log N / \log \log N} \leq \frac{C^{1/p} |\log p| \log 2}{2p},$$

and hence we get the desired asymptotics when  $p \nearrow 1$  with  $c = (C \log 2)/2$ .  $\square$

*Proof of the lower bound in Theorem 5.3.* We consider first the special case when  $M$  is the product of the first  $k$  prime numbers,  $M = p_1 \cdots p_k$ . By the prime number theorem, we have  $k \sim \log M / \log \log M$ . We use then the function

$$f_M(s) := \prod_{j=1}^k \left( \sqrt{1-p_j/2} + p_j^{-s} \sqrt{p_j/2} \right)^{2/p}.$$

We recognize each of the factors of this product as the extremal function from Theorem 4.1. Hence  $\|f_M\|_p = 1$  and

$$f_M(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

with

$$a_M = C(1, p)^k = \left( \sqrt{\frac{2}{p}} \left(1 - \frac{p}{2}\right)^{\frac{1}{p} - \frac{1}{2}} \right)^k.$$

Consequently, by the triangle inequality for the  $L^p$  quasi-metric,

$$(62) \quad C(1, p)^{pk} \leq \|S_{M-1} f_M\|_p^p + \|S_M f_M\|_p^p \leq 2 \max(\|S_{M-1} f_M\|_p^p, \|S_M f_M\|_p^p),$$

and therefore at least one of the quasi-norms  $\|S_{M-1}f_M\|_p$  or  $\|S_M f_M\|_p$  is bounded below by

$$\frac{1}{2}C(1, p)^{(1+o(1))\frac{\log M}{\log \log M}}.$$

Suppose now that an arbitrary  $N$  is given. Set  $n_j := p_1 \cdots p_j$  and

$$J := \max\{j : N/n_j \geq n_j + 1\}.$$

It follows that  $\log n_J = (1/2 + o(1)) \log N$ . There are now two cases to consider:

- (1) Suppose  $\|S_{n_J} f_{n_J}\|_p$  is large. We set  $x_N := \lfloor N/n_J \rfloor$  and define

$$g_N(s) := x_N^{-s} f_{n_J}(s).$$

Then  $(S_N g_N)(s) = x_N^{-s} (S_{n_J} f_{n_J})(s)$  because  $x_N = N/n_J - \varepsilon$  for some  $0 \leq \varepsilon < 1$ , and so

$$x_N(n_J + 1) = (N/n_J - \varepsilon)(n_J + 1) = N + N/n_J - \varepsilon(n_J + 1) > N,$$

where we in the last step used the definition of  $J$ .

- (2) Suppose  $\|S_{n_J-1} f_{n_J}\|_p$  is large. We set  $x_N := \lfloor N/n_J \rfloor$  and define  $g_N$  as in the first case.

Then  $(S_N g_N)(s) = x_N^{-s} (S_{n_J-1} f_{n_J})(s)$  because  $x_N = N/n_J + \varepsilon$  for some  $0 \leq \varepsilon < 1$ , and so

$$x_N(n_J - 1) = (N/n_J + \varepsilon)(n_J - 1) = N - N/n_J + \varepsilon(n_J - 1) < N,$$

where we in the last step again used the definition of  $J$ .

In either case, since (62) holds for  $M = n_J$  and  $\log n_J = (1/2 + o(1)) \log N$ , we conclude that

$$\liminf_{N \rightarrow \infty} \frac{\log \|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p}}{\log N / \log \log N} \geq \frac{1}{2} \log C(1, p).$$

The proof is finished by invoking the asymptotic estimate (50). □

Up to the precise values of  $\alpha_p$  and  $\beta_p$ , the problem of estimating  $\|S_N\|_{\mathcal{H}^p \rightarrow \mathcal{H}^p}$  for  $0 < p < 1$  is solved by Theorem 5.3. This result is, however, somewhat deceptive because it is of no help when we need to estimate  $\|S_N f\|_{\mathcal{H}^p}$  for functions  $f$  of number theoretic interest, such as (2). In fact, in that case, Lemma 5.1 gives a much better bound. The problem of estimating such norms (or quasi-norms) is the topic of the final section of this paper.

## 6. PSEUDOMOMENTS OF THE RIEMANN ZETA FUNCTION AND RELATED DIRICHLET SERIES

**6.1. Generalities about moments and pseudomoments of  $\zeta(1/2 + it)$ .** This section is partially motivated by our desire to understand the distribution of large values of the Riemann zeta function  $\zeta(s)$  on the critical line  $\sigma = 1/2$ . We begin by recalling the classical approximation

$$\zeta(\sigma + it) = \sum_{n \leq x} n^{-\sigma - it} - \frac{x^{1-\sigma-it}}{1-\sigma-it} + O(x^{-\sigma}),$$

which holds uniformly in the range  $\sigma \geq \sigma_0 > 0$ ,  $|t| \leq x$  (see [51, Thm. 4.11]). This means that

$$\left| \zeta(1/2 + it) - \sum_{n \leq 2T} n^{-1/2-it} \right| = O(T^{-1/2}), \quad T \leq t \leq 2T,$$

and so our problem is about the size of  $\sum_{n \leq 2T} n^{-1/2-it}$  on the interval  $[T, 2T]$ .

We recall briefly some known facts about the distribution of  $|\zeta(1/2 + it)|$  on  $[T, 2T]$ . First, by a celebrated result of Selberg (see [46, 48]),  $\log |\zeta(1/2 + it)|$  has an approximate normal distribution with mean zero and variance  $\frac{1}{2} \log \log T$  on  $[T, 2T]$ . This implies that a “typical” value of  $|\zeta(1/2 + it)|$  and hence of  $|\sum_{n \leq 2T} n^{-1/2-it}|$  on  $[T, 2T]$  is  $e^{\sqrt{(1/2) \log \log T}}$ . More precise information

about the distribution of  $|\zeta(1/2 + it)|$  can be acquired from the size of the moments. One expects that

$$M_k(T) := \frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)|^{2k} dt \sim A_k (\log T)^{k^2}$$

for some constant  $A_k$  for which one even has precise predictions [16]. This asymptotic behavior is known to hold when  $k = 1, 2$  by results of respectively Hardy and Littlewood [22] and Ingham [34]. An unconditional lower bound  $M_k(T) \gg (\log T)^{k^2}$  is known in the range  $k \geq 1$  [38], and this is known to hold conditionally for all  $k > 0$  by work of Ramachandra (see [39, 40]) and Heath-Brown [27]. Harper [24], building and improving on

work of Soundararajan [49], showed that the upper bounds of optimal order  $M_k \ll (\log T)^{k^2}$  also hold conditionally for all  $k > 0$ .

By the Bohr correspondence, we may think of the interval  $[T, 2T]$  as a subset of  $\mathbb{T}^\infty$ , and an interesting question is then to understand the distribution of  $|\sum_{n \leq 2T} n^{-1/2-it}|$  on the entire torus  $\mathbb{T}^\infty$  and, in particular, to compare with what we have on the subset  $[T, 2T]$ . We use again the notation  $Z_N(s) := \sum_{n \leq N} n^{-1/2-s}$  and, following Conrey and Gamburd [16], refer to the corresponding moments

$$\Psi_k(N) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_T^{2T} |Z_N(it)|^{2k} dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |Z_N(it)|^{2k} dt = \|Z_N\|_{\mathcal{H}^{2k}}^{2k}$$

as the pseudomoments of  $\zeta(s)$ . Conrey and Gamburd found that

$$(63) \quad \Psi_k(N) = C_k (\log N)^{k^2} + O((\log N)^{k^2-1})$$

when  $k$  is an integer, and a precise value for the constant  $C_k$  was given (see the next subsection). For general  $k > 0$ , one may expect a similar behavior. To this end, it is known from [8] that

$$(64) \quad \Psi_k(N) \asymp_k (\log N)^{k^2}, \quad k > 1/2$$

and that

$$(65) \quad \Psi_k(N) \gg_k (\log N)^{k^2}, \quad k > 0.$$

However, we know only that  $\Psi_{1/2}(N) \ll (\log \log N)(\log N)^{1/4}$  and that

$$\Psi_k(N) \ll_k (\log N)^{k/2}, \quad 0 < k < 1/2.$$

Here the upper bounds are established by Helson's theorem on the partial sum operator, and the lower bounds are deduced from Hardy–Littlewood inequalities. We refer to [8] for the details.

There are several remaining problems. The most obvious of these is to get a better upper bound when  $0 < k \leq 1/2$ . Another problem, to be considered next, is to sharpen the asymptotic bounds in (64); we will obtain fairly precise bounds on the implied constants in this relation.

Unfortunately, we have not been able to improve the estimates in the range  $0 < k \leq 1/2$ . Instead, we have considered the closely related problem of the pseudomoments of  $\zeta^\alpha(s)$  for  $\alpha > 1$ . The somewhat surprising conclusion is that, in this case, the lower bound obtained from the Hardy–Littlewood inequality (the “multiplicative bound”) does not give the right asymptotic order for small  $k$ .

**6.2. Bounds for the pseudomoments of  $\zeta(1/2 + it)$  for  $k \geq 1$ .** For  $k$  a positive integer, Conrey and Gamburd [16] computed the constant  $C_k$  in (63): They found that  $C_k = a_k \gamma_k$ , where  $a_k$  is an arithmetic factor defined by

$$a_k := \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{j=0}^{\infty} \frac{c_k^2(j)}{p^j}$$

and  $\gamma_k$  is a geometric factor (the volume of a convex polytope). Bondarenko, Heap, and Seip [8] investigated the asymptotic behavior of  $\Psi_k(N)/(\log N)^{k^2}$  and found a lower bound of super-exponential decay using (35) and an upper bound of super-exponential growth using Helson's theorem for the partial sum operator.

From the result in [16] one suspects that super-exponential decay is correct, and this was conjectured in [8, Sec. 5]. We will now verify that for  $k \geq 1$ , the lower bound is indeed of the correct order. We will do this by replacing the estimates for the partial sum operator with Theorem 3.4. We also include additional details in the computation of the lower estimate from [8] to obtain an explicit lower bound for comparison.

**Theorem 6.1.** *Suppose that  $k \geq 1$ . Then*

$$\begin{aligned} \frac{\Psi_k(N)}{(\log N)^{k^2}} &\leq \frac{1}{\Gamma(k+1)^k} \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \left(1 - \frac{k}{[k]} \frac{1}{p}\right)^{-k[k]}, \\ \frac{\Psi_k(N)}{(\log N)^{k^2}} &\geq \frac{1}{\Gamma([2k]k+1)^{\frac{k}{[2k]}}} \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \left(1 + [2k]k \frac{1}{p}\right)^{\frac{k}{[2k]}}. \end{aligned}$$

In particular, as  $k \rightarrow \infty$ , we get that

$$(66) \quad \exp((-2 + o(1))k^2 \log k) \leq \frac{\Psi_k(N)}{(\log N)^{k^2}} \leq \exp((-1 + o(1))k^2 \log k).$$

It is interesting to observe the similarity between the lower bound in (66) and the unconditional bound

$$M_k(T) \geq \exp((-2 + o(1))k^2 \log k) (\log T)^2$$

obtained by Radziwiłł and Soundararajan [38]. Likewise, we observe that the upper bound in (66) is in agreement with the expected behavior

$$M_k(T) \sim \exp((-1 + o(1))k^2 \log k) (\log T)^2,$$

conjectured by Conrey and Gonek [17].

*Proof of the upper estimate in Theorem 6.1.* Inserting  $Z_N$  into (33), we get

$$\Psi_k(N) = \|Z_N\|_{\mathcal{H}_{2k}^{2k}} \leq \left( \sum_{n=1}^N \frac{d_{[k]}(n)}{n} \left(\frac{k}{[k]}\right)^{\Omega(n)} \right)^k.$$

Using Lemma 3.6 and Abel summation, we find that

$$\sum_{n=1}^N \frac{d_{[k]}(n)}{n} \left(\frac{k}{[k]}\right)^{\Omega(n)} = \frac{\mathcal{G}_k(1)}{\Gamma(k+1)} (\log N)^k + O((\log N)^{k-1}).$$

We complete the proof by inspecting the Euler product for  $\mathcal{G}_k(1)$  and (39). For the asymptotic estimate, we may safely assume  $k \geq 2$ , in which case Lemma 3.5 gives  $\mathcal{G}_k(1) \asymp 1$ . Hence the main



contribution to the decay comes from the Gamma function, and the desired result follows from Stirling's formula:

$$\Gamma(k+1)^k = \exp((1+o(1))k^2 \log k). \quad \square$$

The following argument can be extracted from [8, pp. 201–202], but we include some details here for the reader's benefit.

*Proof of the lower estimate in Theorem 6.1.* We want to use (35), but  $k = p/2 \geq 1$ . To remedy this, we write  $2k = \ell r$  where  $\ell \geq [2k]$  is an integer to be chosen later that ensures that  $r < 2$ . Note that if  $n \leq N$ , then

$$\left| \frac{|\mu(n)|}{d_{2/r}(n)} \right| \sum_{\substack{n_1 \cdots n_\ell = n \\ n_1, \dots, n_\ell \leq N}} \frac{1}{\sqrt{n_1}} \cdots \frac{1}{\sqrt{n_\ell}} \Big|^2 = \frac{|\mu(n)|}{d_{2/r}(n)} \frac{d_\ell^2(n)}{n} = \frac{|\mu(n)|}{n} d_{\ell k}(n).$$

Using (35) and removing all terms in the sum for which  $N < n \leq N^\ell$ , we get the lower bound

$$\|Z_N\|_{2k}^{2k} = \|Z_N^\ell\|_r^r \geq \left( \sum_{n=1}^N \frac{|\mu(n)|}{n} d_{\ell k}(n) \right)^{\frac{k}{\ell}}.$$

As above, one checks that

$$\sum_{n=1}^N \frac{|\mu(n)|}{n} d_{\ell k}(n) = \tilde{C}_k (\log N)^{\ell k} + O((\log N)^{\ell k-1})$$

with

$$(67) \quad \tilde{C}_k = \frac{1}{\Gamma(\ell k + 1)} \prod_p \left( 1 - \frac{1}{p} \right)^{\ell k} \left( 1 + \frac{\ell k}{p} \right).$$

The asymptotic behavior of the Euler product in (67) has been estimated in [8, p. 202], where it was found that

$$\prod_p \left( 1 - \frac{1}{p} \right)^{\ell k} \left( 1 + \frac{\ell k}{p} \right) = \exp((-1+o(1))\ell k \log \log(\ell k)).$$

Therefore the decay is again controlled by  $\Gamma(\ell k + 1)^{k/\ell}$ . Clearly, choosing  $\ell$  as small as possible is optimal, and we therefore set  $\ell = [2k]$ . The proof is completed by similar considerations as in the preceding argument.  $\square$

Theorem 3.4 allows us to improve the more general results of [8] concerning Dirichlet series of the form

$$F(s) = \sum_{n=1}^{\infty} \psi(n) n^{-1/2-s}$$

for a suitable multiplicative function  $\psi(n)$ , in the same way as done above for the Riemann zeta function, as well as to relax the presumed growth condition on  $\psi$  for  $k \geq 3$ . Since the computations go through as before, we refrain from carrying out the details.

**6.3. Pseudomoments of  $\zeta^\alpha(s)$  for  $\alpha > 1$  and small  $k$ .** We define the pseudomoments of  $\zeta^\alpha(s)$  as  $\Psi_{k,\alpha}(N) := \|Z_{N,\alpha}\|_{\mathcal{H}^{2k}}^{2k}$ , where

$$Z_{N,\alpha}(s) := \sum_{n \leq N} d_\alpha(n) n^{-s-1/2}.$$

Letting  $F_N$  be as defined in (2), we see that then  $Z_{N,\alpha} = (S_N F_N^\alpha)(s)$ . We know from [8] that these pseudomoments satisfy the relation

$$(68) \quad \Psi_{k,\alpha}(N) \asymp (\log N)^{k^2 \alpha^2}$$

when  $k > 1/2$ . We will now show that this result fails for small  $k < 1/2$  when  $\alpha > 1$ . If we agree that the moments of  $\zeta^\alpha(s)$  are just the moments of  $|\zeta(s)|^\alpha$ , then we see that our result implies that, on the Riemann hypothesis, there is a discrepancy between the behavior of the pseudomoments and the moments of  $\zeta^\alpha(s)$  for small  $k$  when  $\alpha > 1$ .

**Theorem 6.2.** *Suppose that  $\alpha \geq 1$ . For every  $k > 0$ , there exists a constant  $c(k)$  such that*

$$\Psi_{k,\alpha}(N) \gg (\log N)^{k \log \alpha^2} \exp\left(-c(k) \sqrt{\log \log N \log \log \log N}\right)$$

*holds for arbitrarily large  $N$ .*

This is incompatible with (68) when  $\alpha > 1$  and  $k < (\log \alpha^2)/\alpha^2$ . From this we observe that, whenever  $k < 1/e$ , we can find  $\alpha > 1$  such that (68) fails.

We prepare for the proof of Theorem 6.2 by establishing two lemmas.

**Lemma 6.3.** *Suppose that  $\alpha \geq 1$ . Then*

$$\mathbb{E} \left| \sum_{M/2 < n \leq M} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right| \gg (\log \log M)^{-3+o(1)},$$

*where the implicit constant  $o(1)$  depends only on  $M$ .*

Here we apply the probabilistic notation of Subsection 2.4. We defer the proof of Lemma 6.3 until the end of this subsection.

Our second lemma is a result on the distribution of

$$N(x, m) := \sum_{n \leq x, \Omega(n)=m} 1,$$

similar in spirit to the Erdős–Kac theorem, saying that  $N(x, m)$  is mainly concentrated on the set

$$I_C := \left[ \log \log x - C \sqrt{\log \log x \log \log \log x}, \log \log x + C \sqrt{\log \log x \log \log \log x} \right]$$

when  $x$  is large and  $C$  is a suitable positive constant. To deduce this result, we rely on an estimate of Sathe (see [47]) saying that

$$(69) \quad N(x, m) \leq C \frac{x}{\log x} \frac{(\log \log x)^{m-1}}{(m-1)!}$$

whenever  $x > 10$  and  $1 \leq m \leq (3/2) \log \log x$ , with  $C$  an absolute constant. Choosing  $C$  large enough and using Stirling's formula, we therefore find that

$$(70) \quad \sum_{m \leq (3/2) \log \log x, m \notin I_C} N(x, m) \leq \frac{x}{2(\log \log x)^8}$$

when  $x$  is sufficiently large. Using instead of (69) formula (7) from [33], we deduce that

$$(71) \quad \sum_{m \geq (3/2) \log \log x} N(x, m) \leq \frac{x}{(\log x)^{1/100}}$$

for  $x$  large enough. Combining (70) and (71), we obtain the following.

**Lemma 6.4.** *There exists an absolute constant  $C > 0$  such that*

$$\sum_{m \notin I_C} N(x, m) \leq \frac{x}{(\log \log x)^8}$$

for all sufficiently large  $x$ .

*Proof of Theorem 6.2.* We write

$$D_{N,\alpha}(s) := \sum_{N/2 < n \leq N} d_\alpha(n) \alpha^{-\Omega(n)} n^{-s-1/2}$$

so that

$$Z_{N,\alpha}(s) - Z_{N/2,\alpha}(s) = \sum_{m \geq 0} \alpha^m P_m D_{N,\alpha}(s).$$

By Lemma 4.4, we have for every  $m$  and  $0 < q < 1$

$$(72) \quad \|Z_{N,\alpha} - Z_{N/2,\alpha}\|_q \gg \alpha^m m^{1-1/q} \|P_m D_{N,\alpha}\|_q.$$

We will combine (72) with an estimate that we obtain from the two lemmas above.

In what follows, we will use that the  $L^2$  norm of  $D_{N,\alpha}$  can be estimated in a trivial way because  $d_\alpha(n) \alpha^{-\Omega(n)} \leq 1$ . First, applying Hölder's inequality in the form

$$\|f\|_1^{2-q} \leq \|f\|_q^q \|f\|_2^{2-2q}$$

along with Lemma 6.3 and a trivial  $L^2$  estimate, we find that

$$\left\| \sum_{m \geq 0} P_m D_{N,\alpha} \right\|_q^q \gg (\log \log N)^{-6+o(1)}$$

whenever  $0 < q < 1$ . Using the triangle inequality for the  $L^q$  quasi-norm and the trivial bound  $\|f\|_q \leq \|f\|_2$ , we obtain from this that

$$\sum_{m \in I_C} \|P_m D_{N,\alpha}\|_q^q + \left\| \sum_{m \notin I_C} P_m D_{N,\alpha} \right\|_2^q \gg (\log \log N)^{-6+o(1)}.$$

Hence, by a trivial  $L^2$  bound and an application of Lemma 6.4, there exists a constant  $C$  such that

$$\sum_{m \in I_C} \|P_m D_{N,\alpha}\|_q^q \gg (\log \log N)^{-6+o(1)}.$$

Thus, since  $|I_C| = O(\sqrt{\log \log N \log \log \log N})$ , there exists an  $m$  satisfying

$$\log \log N - C \sqrt{\log \log N \log \log \log N} \leq m \leq \log \log N + C \sqrt{\log \log N \log \log \log N}$$

such that

$$(73) \quad \|P_m D_{N,\alpha}\|_q^q \geq (\log \log N)^{-6.5+o(1)}.$$

We now set  $q = 2k$ . Combining (72) and (73), we find that for some  $c(k, \alpha)$

$$\|Z_{N,\alpha} - Z_{N/2,\alpha}\|_{2k}^{2k} \gg (\log N)^{k \log \alpha^2} \exp\left(-c(k, \alpha) \sqrt{\log \log N \log \log \log N}\right).$$

Since

$$\|Z_{N,\alpha} - Z_{N/2,\alpha}\|_{2k}^{2k} \leq \|Z_{N,\alpha}\|_{2k}^{2k} + \|Z_{N/2,\alpha}\|_{2k}^{2k},$$

this means that at least one of the pseudomoments  $\Psi_{k,\alpha}(N/2)$  or  $\Psi_{k,\alpha}(N)$  satisfies the lower bound asserted by the theorem.  $\square$

*Proof of Lemma 6.3.* Let  $N_x$  be the set of  $x$ -smooth numbers, i.e.,

$$N_x := \{n \in \mathbb{N} : p \text{ a prime such that } p|n \Rightarrow p \leq x\}.$$

We start with the following identity which holds for every real  $t$ :

$$(74) \quad \int_1^\infty \frac{\sum_{y/2 < n \leq y, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2}}{y^{1+1/\log x + it}} dy = \left( \frac{1 - 2^{-1/\log x - it}}{1/\log x + it} \right) \times \sum_{n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2 - 1/\log x - it}.$$

Our first goal is to estimate the supremum of the right hand side in (74) for  $t$  from a reasonably short interval. We have

$$(75) \quad \left| \sum_{n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2 - 1/\log x - it} \right| = \prod_{p \leq x} \left| 1 + \sum_{j=1}^\infty c_\alpha(j) \alpha^{-j} z(p)^j p^{-j(1/2 + 1/\log x + it)} \right| \\ \asymp \exp \left( \Re \left( \sum_{p \leq x} z(p) p^{-1/2 - 1/\log x - it} \right) + \frac{1}{2\alpha} \Re \left( \sum_{p \leq x} z(p)^2 p^{-1 - 2/\log x - 2it} \right) \right)$$

for all points of the configuration space  $(z(p))_{p \leq x}$ . As in [26, Lem. 1], we can modify the proof of [25, Cor. 2] to show that

$$\sup_{1 \leq t \leq 2(\log \log x)^2, |1 - 2^{-it}| \geq 1/4} \left( \Re \left( \sum_{p \leq x} z(p) p^{-1/2 - 1/\log x - it} \right) + \frac{1}{2\alpha} \Re \left( \sum_{p \leq x} z(p)^2 p^{-1 - 2/\log x - 2it} \right) \right) \\ \geq \log \log x - \log \log \log x + O((\log \log \log x)^{3/4})$$

with probability  $1 - o(1)$  as  $x \rightarrow \infty$ . To achieve this, we add a minor technical detail: In the part of the argument that follows [25, Sec. 6], we only take into account those integers  $n$ ,  $1 \leq n \leq (\log \log x)^2$ , such that

$$\min_{2n+1 \leq t \leq 2n+2} |1 - 2^{-it}| \geq 1/4,$$

noting that the number of such  $n$  is bounded below by a constant times  $(\log \log x)^2$ . Combining the latter inequality with (75), we obtain that with probability  $1 - o(1)$

$$\sup_{1 \leq t \leq 2(\log \log x)^2, |1 - 2^{-it}| \geq 1/4} \left| \sum_{n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2 - 1/\log x - it} \right| \geq \log x (\log \log x)^{-1+o(1)}.$$

Now taking the supremum of the absolute value of both sides in (74), we find that

$$\int_1^\infty \frac{\left| \sum_{y/2 < n \leq y, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right|}{y^{1+1/\log x}} dy \geq \log x (\log \log x)^{-3+o(1)}$$

with probability  $1 - o(1)$ . Hence taking the expectation over the entire configuration space  $(z(p))_{p \leq x}$ , we finally obtain that, say for all  $x > 3$ ,

$$(76) \quad \int_1^\infty \frac{\mathbb{E} \left| \sum_{y/2 < n \leq y, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right|}{y^{1+1/\log x}} dy \geq \log x (\log \log x)^{-3+o(1)}.$$

Now we will show that the assertion of the lemma follows from (76). To this end, we begin by fixing a positive integer  $M$ . We will use (76) for  $x$  such that  $M = x^{10 \log \log \log x}$ . Applying the

Cauchy–Schwarz inequality in the form  $(\mathbb{E}|X|)^2 \leq \mathbb{E}|X|^2$  and recalling that  $d_\alpha(n)\alpha^{-\Omega(n)} \leq 1$ , we find that

$$\int_{\sqrt{M}}^{\infty} \mathbb{E} \left| \frac{\sum_{y/2 < n \leq y, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2}}{y^{1+1/\log x}} \right| dy \leq \sqrt{2} \int_{\sqrt{M}}^{\infty} \frac{1}{y^{1+1/\log x}} dy = \sqrt{2} \log x (\log \log x)^{-5}.$$

Combining this bound with (76), we find that

$$(77) \quad \int_1^{\sqrt{M}} \mathbb{E} \left| \frac{\sum_{y/2 < n \leq y, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2}}{y^{1+1/\log x}} \right| dy \geq \log x (\log \log x)^{-3+o(1)},$$

which is the relation to be used below.

Set  $S_{M,\alpha}(z) := \sum_{M/2 < n \leq M} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2}$ , and let  $N_x^\perp$  be the set of integers with prime divisors that are all larger than  $x$ . Write

$$S_{M,\alpha}(z) = \sum_{n \in N_x^\perp, 1 \leq n \leq M} c_n d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2},$$

where

$$c_y := \sum_{k \in N_x, M/(2y) \leq k \leq M/y} d_\alpha(k) \alpha^{-\Omega(k)} z(k) k^{-1/2}.$$

By Helson's inequality (31), we find that

$$(78) \quad \mathbb{E}|S_{M,\alpha}| \geq \mathbb{E} \left( \sum_{n \in N_x^\perp, 1 \leq n \leq M} \frac{|c_n|^2 d_\alpha(n)^2 \alpha^{-2\Omega(n)}}{d(n)n} \right)^{1/2} \geq \mathbb{E} \left( \sum_{x < p \leq M} \frac{|c_p|^2}{2p} \right)^{1/2}.$$

We now want to relate the right-hand side of (78) to the integral

$$\int_x^M \left| \sum_{M/(2y) \leq n \leq M/y, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right|^2 \frac{dy}{y} = \int_x^M |c_y|^2 \frac{dy}{y}.$$

To this end, we begin by considering a short interval  $[\xi, \xi + \xi^\delta] \subset [x, M]$ , where  $7/12 < \delta < 1$  is a fixed parameter. If  $\xi$  is sufficiently large, then by [28], this interval contains at least  $\xi^\delta/(2 \log \xi)$  primes. We partition accordingly the interval into  $\lceil \xi^\delta/(2 \log \xi) \rceil$  subintervals of equal length  $\xi^\delta/\lceil \xi^\delta/(2 \log \xi) \rceil$ . We make a one-to-one correspondence between these subintervals and the first  $\lceil \xi^\delta/(2 \log \xi) \rceil$  primes in  $[\xi, \xi + \xi^\delta]$ , and hence we associate with every  $y$  in  $[\xi, \xi + \xi^\delta]$  a prime  $p = p(y)$  that is also in  $[\xi, \xi + \xi^\delta]$ . We write  $\tilde{c}_y := c_y - c_{p(y)}$  and notice that

$$|c_y|^2 \leq 2(|c_{p(y)}|^2 + |\tilde{c}_y|^2),$$

where  $\mathbb{E}|\tilde{c}_y|^2 \ll \xi^{\delta-1} \ll y^{\delta-1}$ . From this we get that

$$\int_\xi^{\xi+\xi^\delta} \left| \sum_{M/(2y) \leq n \leq M/y, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right|^2 \frac{dy}{y} \ll (\log \xi) \sum_{\xi \leq p \leq \xi+\xi^\delta} \frac{|c_p|^2}{p} + \int_\xi^{\xi+\xi^\delta} \frac{|\tilde{c}_y|^2}{y} dy.$$

Repeating this construction and summing over a suitable collection of intervals  $[\xi, \xi + \xi^\delta]$ , we then obtain

$$\sum_{x < p \leq M} \frac{|c_p|^2}{p} + \int_x^M \frac{|\tilde{c}_y|^2}{y} dy \gg \frac{1}{\log M} \int_x^M \left| \sum_{M/(2y) \leq n \leq M/y, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right|^2 \frac{dy}{y}.$$

By the change of variables  $u = M/y$  in the integral on the right-hand side and using that  $\log M = 10 \log x \log \log \log x$ , we now deduce that

$$(79) \quad \left( \sum_{x < p \leq M} \frac{|c_p|^2}{p} + \int_x^M \frac{|\tilde{c}_y|^2}{y} dy \right)^{1/2} \\ \gg \left( \frac{1}{\log x \log \log \log x} \int_1^{M/x} \left| \sum_{u/2 \leq n \leq u, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right|^2 \frac{du}{u} \right)^{1/2}.$$

We are now ready to finish the proof by putting our three basic estimates (77), (78), and (79) together. First, by the Cauchy–Schwarz inequality, we have

$$\left( \int_1^{M/x} \left| \sum_{u/2 \leq n \leq u, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right|^2 \frac{du}{u} \right)^{1/2} \left( \int_1^{M/x} \frac{1}{u^{1+2/\log x}} du \right)^{1/2} \\ \geq \int_1^{\sqrt{M}} \frac{\left| \sum_{u/2 \leq n \leq u, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right|}{u^{1+1/\log x}} du.$$

Therefore, taking expectation in (79) and applying (78) together with (77), we find that

$$\mathbb{E}|S_M| \gg \mathbb{E} \left| \left( \sum_{x < p \leq M} \frac{|c_p|^2}{p} + \int_x^M \frac{|\tilde{c}_y|^2}{y} dy \right)^{1/2} \right| - \mathbb{E} \left| \left( \int_x^M \frac{|\tilde{c}_y|^2}{y} dy \right)^{1/2} \right| \\ \gg \mathbb{E} \left| \left( \sum_{x < p \leq M} \frac{|c_p|^2}{p} + \int_x^M \frac{|\tilde{c}_y|^2}{y} dy \right)^{1/2} \right| - x^{-(1-\delta)/2} \\ \gg \frac{1}{\log x (\log \log \log x)^{1/2}} \int_1^{\sqrt{M}} \frac{\mathbb{E} \left| \sum_{u/2 \leq n \leq u, n \in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \right|}{u^{1+1/\log x}} du - x^{-(1-\delta)/2} \\ \geq (\log \log x)^{-3+o(1)} \geq (\log \log M)^{-3+o(1)},$$

and hence the desired estimate has been established.  $\square$

## REFERENCES

1. A. Aleman, J.-F. Olsen, and E. Saksman, *Fatou and brothers Riesz theorems in  $\mathbb{T}^\infty$* , to appear in J. Anal. Math. (arXiv:1512.01509).
2. ———, *Fourier multipliers for Hardy spaces of Dirichlet series*, Int. Math. Res. Not. IMRN (2014), no. 16, 4368–4378.
3. V. V. Arestov, *Integral inequalities for trigonometric polynomials and their derivatives*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 1, 3–22, 239.
4. R. Balasubramanian, B. Calado, and H. Queffélec, *The Bohr inequality for ordinary Dirichlet series*, Studia Math. **175** (2006), no. 3, 285–304.
5. F. Bayart, *Hardy spaces of Dirichlet series and their composition operators*, Monatsh. Math. **136** (2002), no. 3, 203–236.
6. F. Bayart and O. F. Brevig, *Composition operators and embedding theorems for some function spaces of Dirichlet series*, arXiv:1602.03446.
7. F. Bayart, H. Queffélec, and K. Seip, *Approximation numbers of composition operators on  $H^p$  spaces of Dirichlet series*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 2, 551–588.
8. A. Bondarenko, W. Heap, and K. Seip, *An inequality of Hardy–Littlewood type for Dirichlet polynomials*, J. Number Theory **150** (2015), 191–205.
9. A. Bondarenko and K. Seip, *Helson’s problem for sums of a random multiplicative function*, Mathematika **62** (2016), no. 1, 101–110.

10. O. F. Brevig, *An embedding constant for the Hardy space of Dirichlet series*, to appear in Proc. Amer. Math. Soc. (arXiv:1606.03101).
11. O. F. Brevig, J. Ortega-Cerdà, K. Seip, and J. Zhao, *Contractive inequalities for Hardy spaces*, in preparation.
12. O. F. Brevig, K.-M. Perfekt, and K. Seip, *Volterra operators on Hardy spaces of Dirichlet series*, to appear in J. Reine Angew. Math. (arXiv:1602.04729).
13. O. F. Brevig, K.-M. Perfekt, K. Seip, A. G. Siskakis, and D. Vukotić, *The multiplicative Hilbert matrix*, Adv. Math. **302** (2016), 410–432.
14. J. Burbea, *Sharp inequalities for holomorphic functions*, Illinois J. Math. **31** (1987), no. 2, 248–264.
15. B. J. Cole and T. W. Gamelin, *Representing measures and Hardy spaces for the infinite polydisk algebra*, Proc. London Math. Soc. (3) **53** (1986), no. 1, 112–142.
16. B. Conrey and A. Gamburd, *Pseudomoments of the Riemann zeta-function and pseudomagic squares*, J. Number Theory **117** (2006), no. 2, 263–278.
17. J. B. Conrey and S. M. Gonek, *High moments of the Riemann zeta-function*, Duke Math. J. **107** (2001), no. 3, 577–604.
18. A. Defant and A. Pérez, *Optimal comparison of  $P$ -norms of Dirichlet polynomials*, arXiv:1603.02128.
19. P. L. Duren, *Theory of  $H^p$  spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970.
20. P. L. Duren, B. W. Romberg, and A. L. Shields, *Linear functionals on  $H^p$  spaces with  $0 < p < 1$* , J. Reine Angew. Math. **238** (1969), 32–60.
21. J. B. Garnett, *Bounded analytic functions*, first ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007.
22. G. H. Hardy and J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. **41** (1916), no. 1, 119–196.
23. ———, *Notes on the theory of series (XIII): Some new properties of Fourier constants*, J. London Math. Soc. **S1-6** (1931), no. 1, 3–9.
24. A. J. Harper, *Sharp conditional bounds for moments of the Riemann zeta function*, arXiv:1305.4618.
25. ———, *Bounds on the suprema of Gaussian processes, and omega results for the sum of a random multiplicative function*, Ann. Appl. Probab. **23** (2013), no. 2, 584–616.
26. A. J. Harper, A. Nikeghbali, and M. Radziwiłł, *A note on Helson's conjecture on moments of random multiplicative functions*, Analytic number theory, Springer, Cham, 2015, pp. 145–169.
27. D. R. Heath-Brown, *Fractional moments of the Riemann zeta function*, J. London Math. Soc. (2) **24** (1981), no. 1, 65–78.
28. ———, *The number of primes in a short interval*, J. Reine Angew. Math. **389** (1988), 22–63.
29. H. Hedenmalm, P. Lindqvist, and K. Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in  $L^2(0, 1)$* , Duke Math. J. **86** (1997), no. 1, 1–37.
30. H. Helson, *Conjugate series and a theorem of Paley*, Pacific J. Math. **8** (1958), 437–446.
31. ———, *Hankel forms and sums of random variables*, Studia Math. **176** (2006), no. 1, 85–92.
32. H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. **99** (1958), 165–202.
33. H.-K. Hwang, *Sur la répartition des valeurs des fonctions arithmétiques. Le nombre de facteurs premiers d'un entier*, J. Number Theory **69** (1998), no. 2, 135–152.
34. A. E. Ingham, *Mean-value theorems in the theory of the Riemann zeta-function*, Proc. London Math. Soc. (2) **27** (1926), 273–300.
35. J.-F. Olsen, *Local properties of Hilbert spaces of Dirichlet series*, J. Funct. Anal. **261** (2011), no. 9, 2669–2696.
36. J.-F. Olsen and E. Saksman, *On the boundary behaviour of the Hardy spaces of Dirichlet series and a frame bound estimate*, J. Reine Angew. Math. **663** (2012), 33–66.
37. H. Queffélec and M. Queffélec, *Diophantine approximation and Dirichlet series*, Harish-Chandra Research Institute Lecture Notes, vol. 2, Hindustan Book Agency, New Delhi, 2013.
38. M. Radziwiłł and K. Soundararajan, *Continuous lower bounds for moments of zeta and  $L$ -functions*, Matematika **59** (2013), no. 1, 119–128.
39. K. Ramachandra, *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. I*, Hardy-Ramanujan. J. **1** (1978), 1–15.



40. ———, *On the mean-value and omega-theorems for the Riemann zeta-function*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 85, Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, 1995.
41. W. Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc., New York–Amsterdam, 1969.
42. ———, *Fourier analysis on groups*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1990, Reprint of the 1962 original, A Wiley-Interscience Publication.
43. E. Saksman and K. Seip, *Some open questions in analysis for Dirichlet series*, in “Recent Progress on Operator Theory and Approximation in Spaces of Analytic Functions”, pp. 179–193, Contemp. Math. **679**, Amer. Math. Soc., Providence RI, 2016.
44. ———, *Integral means and boundary limits of Dirichlet series*, Bull. Lond. Math. Soc. **41** (2009), no. 3, 411–422.
45. K. Seip, *Zeros of functions in Hilbert spaces of Dirichlet series*, Math. Z. **274** (2013), no. 3–4, 1327–1339.
46. A. Selberg, *Contributions to the theory of the Riemann zeta-function*, Arch. Math. Naturvid. **48** (1946), no. 5, 89–155.
47. ———, *Note on a paper by L. G. Sathe*, J. Indian Math. Soc. (N.S.) **18** (1954), 83–87.
48. ———, *Old and new conjectures and results about a class of Dirichlet series*, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), Univ. Salerno, Salerno, 1992, pp. 367–385.
49. K. Soundararajan, *Moments of the Riemann zeta function*, Ann. of Math. (2) **170** (2009), no. 2, 981–993.
50. G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015.
51. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., The Clarendon Press, Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown.
52. M. von Golitschek and G. G. Lorentz, *Bernstein inequalities in  $L_p$ ,  $0 \leq p \leq +\infty$* , Rocky Mountain J. Math. **19** (1989), no. 1, 145–156, Constructive Function Theory—86 Conference (Edmonton, AB, 1986).
53. F. B. Weissler, *Logarithmic Sobolev inequalities and hypercontractive estimates on the circle*, J. Funct. Anal. **37** (1980), no. 2, 218–234.

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